# A pen-picture of optical solitons for the concatenation model with power-law of self-phase modulation by Sardar's sub-equation method and tanh-coth approach

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This work is about the retrieval of optical solitons for the concatenation model with power-law of self-phase modulation by the aid of Sardar's subequation approach and its variation. The tanh-coth approach reveals the dark-singular straddled optical solitons solutions to the model. The parameter constraints for the existence of such solitons are also enumerated. Numerical simulations are also presented.

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## 1. Introduction

A decade ago, the concatenation model with Kerr law of self—phase modulation (SPM) was first proposed to address the soliton propagation dynamics through optical fibers across trans—continental and trans—oceanic distances [1-5]. This model was formulated by conjoining the three fundamental governing equations that appear in nonlinear fiber optics [6-10]. They are the nonlinear Schrodinger's equation (NLSE), the Lakshmanan—Porsezian—Daniel (LPD) model and the Sasa—Satsuma equation (SSE) [11-15]. Later this model gained extreme popularity that gave way to a plethora of results being reported all across the board [16-20]. A few salient features that have been addressed are the retrieval of conservation laws, Painleve analysis, numerical analysis of the model by the Laplace—Adomian decomposition, addressing the model with the presence of white noise and several other features [21-25]. Subsequently this model was addressed with differential group delay which gave way to additional successful investigations [26-28]. The soliton solutions of the coupled system was later recovered with the usage of several integration algorithms including the method of undetermined coefficients and other such approaches.

The current paper moves further along with the model. The model is now being considered with power—law of SPM. This will be addressed with the Sardar sub—equation approach and its versions. This would lead to a sort of generalized format of 1—soliton solutions. Later, the model will be studied using the tanh—coth approach and this would reveal dark—singular straddled solitons to the model. The parameter constraints for the existence of such solitons that naturally emerged from the solutions are also presented. The surface plots of the solitons are also given just for the complete picture of the solutions structures. The details are exhibited after a revisitation to the model.

## 2. Governing model

The concatenation model with power nonlinearity is formulated as [11]:

$$i\Phi_t + a\Phi_{xx} + b|\Phi|^{2n}\Phi$$
$$+c_1\begin{bmatrix} \sigma_1\Phi_{xxxx} + \sigma_2(\Phi_x)^2\Phi^* + \sigma_3|\Phi_x|^2\Phi \\ +\sigma_4|\Phi|^{2n}\Phi_{xx} + \sigma_5\Phi^2\Phi^*_{xx} + \sigma_6|\Phi|^{2n+2}\Phi \end{bmatrix}$$
$$+ic_2[\sigma_7\Phi_{xxx} + \sigma_8[\Phi]^{2n}\Phi_x + \sigma_9\Phi^2\Phi^*_x] = 0. \quad (1)$$

The dimensionless form of the concatenation model is given by (1) where the independent variables are $x$ and $t$ that represent the spatial and temporal coordinates respectively. The dependent variable $\Phi$ represents the wave amplitude. The coefficient of the first term is $i=\sqrt{-1}$ and this term represents the linear temporal evolution. The coefficient of $a$ is the linear chromatic dispersion while the coefficient of $b$ is the SPM with $n$ being the power—law nonlinearity parameter. The coefficients $c_1$ and $c_2$ are the LPD model and SSE respectively, where as the first three terms constitute the NLSE with power—law of SPM. Notably, Eq. (1) collapses to the well-known NLSE with power—law of SPM for $c_1=c_2=0$.

## 3. Travelling waves

The solutions of Eq. (1) supposed as [29-38]

$$\Phi(x,t)=\ u(\xi)\ e^{i\,\theta(x,t)}, \tag{2}$$

where $\xi=(x-\gamma\,t)$ and the phase $\theta(x,t)=-kx+\omega\,t+\theta_0$, $u\,(\xi)$ is the amplitude components of the wave and $\gamma$ is its speed, $k$ is the Soliton frequency, ω is its wavenumber and $\theta_0$ is the phase constant. Using Eq. (2) and their derivatives, Eq.(1) transform to

$$\begin{gathered}[-i\,\gamma\,u'-\omega\,u]+a[u''-2\,i\,k\,u'-k^2\,u]+bu^{2n+1}\\ +c_{\,1}\,\sigma_1(u''''-4\,i\,k\,u'''-6k^2\,u''+4\,ik^3\,u'+k^4\,u)\\ +c_{\,1}(\sigma_2+\sigma_3)\big(u\,u'^2-2u'ki\,u^2-k^2\,u^3\big)\\ +c_{\,1}\sigma_4u^{2n}\,(u''-2\,i\,k\,u'-k^2u)\\ +c_{\,1}\sigma_5\,(u^2u''-2\,i\,ku^2\,u'-k^2u^3)\\ +c_{\,1}\sigma_6u^{2n+3}+c_2\sigma_7(iu'''+3\,k\,u''-3i\,k^2\,u'-\ k^3\,u)\\ +c_2\sigma_8u^{2n}(i\,u'+k\,u)+c_2\sigma_9u^2(i\,u'+k\,u)=0.\end{gathered} \tag{3}$$

Eqs. (3) can be decomposing into real and imaginary parts are respectively expressed as

$$\begin{gathered}c_{\,1}\,\sigma_1u^{(4)}+[a+3c_2\sigma_7\,k\ -6k^2c_{\,1}\,\sigma_1]\,u''\\ +c_{\,1}\sigma_5\,u^2u''+c_{\,1}\sigma_4u^{2n}u''+c_{\,1}(\sigma_2+\sigma_3)u\,u'^2\\ +[c_{\,1}\,\sigma_1k^4\ -ak^2-c_2\sigma_7\,k^3-\omega\,]\,u\\ +bu^{2n+1}+kc_2\sigma_8u^{2n+1}\\ +[kc_2\sigma_9-c_{\,1}(\sigma_2+\sigma_3+\sigma_5)k^2\,]u^3\\ -c_{\,1}\sigma_4u^{2n+1}+c_{\,1}\sigma_6u^{2n+3}=0,\end{gathered} \tag{4}$$

and

$$\begin{gathered}[(c_2\sigma_7-4k\,c_{\,1}\sigma_1)\,u''']\\ +[4\,k^3c_{\,1}\,\sigma_1-3\,k^2c_2\sigma_7-\gamma-2ak]\,u'\\ +[c_2\sigma_9-2k\,c_{\,1}(\sigma_2+\sigma_3+\sigma_5)]u'\,u^2\\ +[c_2\sigma_8-2k\,c_{\,1}\sigma_4]u'u^{2n}\ =0.\end{gathered} \tag{5}$$

From Eq. (5), the soliton speed is:

$$\gamma=-2k(4\,k^2c_{\,1}\,\sigma_1+a), \tag{6}$$

whenever

$$c_2\sigma_9=2k\,c_{\,1}(\sigma_2+\sigma_3+\sigma_5), \tag{7}$$

$$c_2\sigma_8=2k\,c_{\,1}\sigma_4, \tag{8}$$

$$c_2\sigma_7=4\,k\,c_{\,1}\,\sigma_1. \tag{9}$$

From Eq. (8) and Eq. (9):

$$\sigma_4\sigma_7=2\sigma_1\sigma_8. \tag{10}$$

Eq. (4) can be written as

$$\begin{gathered}c_{\,1}\,\sigma_1u^{(4)}+\beta_1\,u''+c_{\,1}\sigma_5\,u^2u''+c_{\,1}\sigma_4u^{2n}u''\\ +c_{\,1}(\sigma_2+\sigma_3)u\,u'^2+\beta_2\,u\\ +\beta_3\,u^3+\beta_4u^{2n+1}+c_{\,1}\sigma_6u^{2n+3}=0,\end{gathered} \tag{11}$$

where

$$\beta_1=a+6\,k^2c_1\,\sigma_1, \tag{12}$$

$$\beta_2=-[ak^2+3\,c_{\,1}\,\sigma_1\,k^4+\omega\,], \tag{13}$$

$$\beta_3=k^2\,c_{\,1}(\sigma_2+\sigma_3+\sigma_5), \tag{14}$$

$$\beta_4=(b+kc_2\sigma_8-c_{\,1}\sigma_4). \tag{15}$$

Setting

$$u=v^{\frac{1}{n}}, \tag{16}$$

Eq. (11) is transformed into

$$c_{\,1}\,\sigma_1\left[\begin{gathered}v^3\ v''''+3\left(\frac{1-n}{n}\right)v^2v'\ v'''\\ +2\left(\frac{1-n}{n}\right)v^2v''^2\\ +5\left(\frac{1-2n}{n}\right)\left(\frac{1-n}{n}\right)v\ v''\,v'^2\\ +\left(\frac{1-3n}{n}\right)\left(\frac{1-2n}{n}\right)\left(\frac{1-n}{n}\right)\,v'^4\end{gathered}\right]$$

$$+\,\beta_1\left[\,v^3\ \ v''+\left(\frac{1-n}{n}\right)v^2\ \ v'^2\right]+\ c_{\,1}\sigma_5\,v^5\ \ v''$$

$$+c_1\sigma_5\left(\frac{1-n}{n}\right)v^{\frac{2}{n}+2}\,v'^2$$

$$+c_1\sigma_4\left[v^5\ v''+\left(\frac{1-n}{n}\right)v^4\,v'^2\right]$$

$$+\frac{c_1(\sigma_2+\sigma_3)}{n}v^{\frac{2}{n}+2}\,v'^2+n^2\beta_2 v^4$$

$$+n^2\beta_3\,v^{\frac{2}{n}+4}+n^2\beta_4 v^6+n^2c_1\sigma_6\,v^{\frac{2}{n}+6}=0. \quad (17)$$

For integrability, the coefficient of $v^{\frac{2}{n}+2}, v^{\frac{2}{n}+4}, v^{\frac{2}{n}+6}$ in Eq. (17) must vanish, thus we obtain the following nonlinear ordinary differential equation:

$$c_1\,\sigma_1\begin{bmatrix} v^3\ v''''+3M_3v^2v'v''' \\ +2M_3v^2v''^2+M_1\,v\ v''\,v'^2+M_2\,v'^4\end{bmatrix}$$

$$+\beta_1[\,v^3\ v''+M_3v^2\,v'^2]+\ c_1\sigma_4\,v^5\ v''$$

$$+c_1\sigma_4M_3\,v^4\,v'^2+n^2\beta_2v^4+n^2\beta_4v^6=0, \quad (18)$$

where

$$\sigma_5=0, \sigma_6=0, (\sigma_2+\sigma_3)=0, \quad (19)$$

$$M_1=\tfrac{5}{n^2}(1-3n+2n^2\,), \quad (20)$$

$$M_2=\tfrac{1}{n^3}(1-6n+11n^2-6\,n^3\,), \quad (21)$$

$$M_3=\left(\tfrac{1-n}{n}\right). \quad (22)$$

### 4. Sardar sub-equation method (SSEM)

The main advantage of the SSEM is that it can generate various forms of soliton solutions, from dark, bright and singular to more interesting forms such as mixed dark-bright, dark-singular, bright-singular, mixed singular. In addition, it provides rational, periodic, trigonometric and other solutions. In this method, to solve Eq. (18) we assume that the solution is of the form [12, 13]

$$v\,(\xi)=\textstyle\sum_{n=0}^{\mathrm{N}}\lambda_n\Psi^n(\xi)\,, \qquad \lambda_{\mathrm{N}}\neq 0\,, \quad (23)$$

where $\lambda_r$ $(n=0,1,\dots,N)$ is a constant to be calculated later. The integer number $N$ is determined by means of the homogeneous balance method principle between the nonlinear term and the highest-order derivative in Eq. (18). Besides, the function $\Psi^r(\xi)$ in Eq. (23) must satisfies the following equation

$$\Psi'(\xi)=\sqrt{\eta_2\Psi(\xi)^4+\eta_1\Psi(\xi)^2+\eta_0}, \quad (24)$$

with $\eta_l, l=0,1,2,$ are constants. Correspondingly, according to the values of the parameters $\eta_l$, Eq. (1) have various known solutions, which are given below [12, 13]:

**Case 1.** $\eta_0=0, \eta_1>0$ and $\eta_2\neq 0$, then

$$\Psi_1^{\pm}(\xi)=\pm\sqrt{-pq\,{}^{\eta_1}/_{\eta_2}}\,\mathrm{sech}_{pq}\big(\sqrt{\eta_1}\xi\big),\ \ \eta_2<0, \quad (25)$$

$$\Psi_2^{\pm}(\xi)=\pm\sqrt{pq\,{}^{\eta_1}/_{\eta_2}}\,\mathrm{csch}_{pq}\big(\sqrt{\eta_1}\xi\big),\ \ \eta_2>0, \quad (26)$$

where

$$\mathrm{sech}_{pq}\big(\sqrt{\eta_1}\xi\big)=\frac{2}{p\,e^{\sqrt{\eta_1}\xi}+q\,e^{-\sqrt{\eta_1}\xi}},$$

$$\mathrm{csch}_{pq}\big(\sqrt{\eta_1}\xi\big)=\frac{2}{p\,e^{\sqrt{\eta_1}\xi}-q\,e^{-\sqrt{\eta_1}\xi}}. \quad (27)$$

**Case 2.** $\eta_0=\frac{1}{4}\frac{\eta_1^2}{\eta_2}, \eta_2>0$ and $\eta_1<0$ , then

$$\Psi_3^{\pm}(\xi)=\pm\sqrt{-{}^{\eta_1}/_{2\eta_2}}\,\tanh_{pq}\left(\sqrt{-\tfrac{\eta_1}{2}}\,\xi\right), \quad (28)$$

$$\Psi_4^{\pm}(\xi)=\pm\sqrt{-{}^{\eta_1}/_{2\eta_2}}\,\coth_{pq}\left(\sqrt{-\tfrac{\eta_1}{2}}\,\xi\right), \quad (29)$$

$$\Psi_5^{\pm}(\xi)=\pm\sqrt{-{}^{\eta_1}/_{2\eta_2}}$$

$$\times\left(\tanh_{pq}\big(\sqrt{-2\eta_1}\xi\big)\pm i\sqrt{pq}\,\mathrm{sech}_{pq}\big(\sqrt{-2\eta_1}\xi\big)\right), \quad (30)$$

$$\Psi_6^{\pm}(\xi)=\pm\sqrt{-{}^{\eta_1}/_{2\eta_2}}$$

$$\times\left(\coth_{pq}\big(\sqrt{-2\eta_1}\xi\big)\pm\sqrt{pq}\,\mathrm{csch}_{pq}\big(\sqrt{-2\eta_1}\xi\big)\right), \quad (31)$$

$$\Psi_7^{\pm}(\xi)=\pm\frac{1}{2}\sqrt{-{}^{\eta_1}/_{2\eta_2}}$$

$$\times\left(\tanh_{pq}\left(\sqrt{-\tfrac{\eta_1}{8}}\,\xi\right)\pm\coth_{pq}\left(\sqrt{\tfrac{-\eta_1}{8}}\,\xi\right)\right), \quad (32)$$

where

$$\tanh_{pq}\big(\sqrt{\eta_1}\xi\big)=\frac{p\,e^{\sqrt{\eta_1}\xi}-q\,e^{-\sqrt{\eta_1}\xi}}{p\,e^{\sqrt{\eta_1}\xi}+q\,e^{-\sqrt{\eta_1}\xi}},$$

$$\coth_{pq}\big(\sqrt{\eta_1}\xi\big)=\frac{p\,e^{\sqrt{\eta_1}\xi}+q\,e^{-\sqrt{\eta_1}\xi}}{p\,e^{\sqrt{\eta_1}\xi}-q\,e^{-\sqrt{\eta_1}\xi}}. \quad (33)$$

#### 4.1. Application of the modified Sardar sub-equation method

We consider our analysis by using the principle of the homogeneous balance method between the nonlinear term $v^3\,v''''$and the linear term $v^6$ from Eq. (18), then $3\,N+N+4=6N$, which gives $N$ =2. Hence, Eq. (23) becomes

$$v(\xi) = \left(\lambda_0 + \lambda_1 \Psi(\xi) + \lambda_2 \Psi^2(\xi)\right). \quad (34)$$

Substituting Eq. (34) into Eq. (18) and taking account of Eq. (24), we obtain an overdetermined system by gathering and equating the coefficient of the independent functions $\Psi^j(\xi)$, to zero, we arrive to the following set of algebraic system equations (SAE) for the following cases:

**Family I:** ($\eta_0 = 0,\ \lambda_0 = 0,\ \lambda_1 = 0$)

We get

$$c_1 \sigma_1 [16 \eta_1{}^2 + 56 M_3 \eta_1{}^2 + 16 M_1 \eta_1{}^2 + 16 M_2 \eta_1{}^2]$$

$$+4 \eta_1 + 4 M_3 \eta_1 + n^2 \beta_2 = 0,$$

$$c_1 \sigma_1 [120 + 192 M_3 + 24 M_3 + 40 M_1 + 32 M_2] \eta_1$$

$$+6 + 4 M_3 = 0,$$

$$c_1 \sigma_1 [120 + 144 M_3 + 72 M_3 + 24 M_1 + 16 M_2] \eta_2{}^2$$

$$+ (4 c_1 \sigma_4 \eta_1 + 4 c_1 \sigma_4 M_3 \eta_1 + n^2 \beta_4) \lambda_2{}^2 = 0. \quad (35)$$

Solving SAE (35) gives:

**Case I.** $\eta_1 > 0$ and $\eta_2 \neq 0$:

$$\eta_{1,1} = \frac{-n^2}{2 c_1 \sigma_1 (3 n^3 - n^2 - 2n + 2)},$$

$$\eta_{1,2} = -\frac{(n+2) n^2}{4 c_1 \sigma_1 [14 n^3 - 4 n^2 + n + 4]},\ \sigma_4 = 0,$$

$$\lambda_2 = \mp \frac{2 \eta_2}{n^2} \sqrt{-\frac{c_1 \sigma_1 [12 n^3 + 7 n^2 + 6 n + 4]}{n \beta_4}}, \quad (36)$$

where

$$\beta_2 = 0,\ \ \omega = -k^2 (1 + 3 c_1 \sigma_1 k^2).$$

The soliton solutions of (1) corresponding to (36), along with solution (24) are

$$\Phi_1(x,t) = \left[\mp \frac{2 \eta_2}{n^2} \sqrt{-\frac{c_1 \sigma_1 [12 n^3 + 7 n^2 + 6 n + 4]}{n \beta_4}}\ \mathrm{sech}^2{}_{pq}\left(\sqrt{\eta_{1,j}}\xi\right)\right]^{\frac{1}{n}}$$

$$\times\ exp[i(-\kappa x + \omega t + \theta_0)], j = 1, 2, \quad (37)$$

$$\Phi_2(x,t) = \left[\mp \frac{2 \eta_2}{n^2} \sqrt{-\frac{c_1 \sigma_1 [12 n^3 + 7 n^2 + 6 n + 4]}{n \beta_4}}\ \mathrm{csch}^2{}_{pq}\left(\sqrt{\eta_{1,j}}\xi\right)\right]^{\frac{1}{n}}$$

$$\times\ exp[i(-\kappa x + \omega t + \theta_0)], j = 1, 2, \quad (38)$$

where $\beta_4 < 0$. Solutions (37) and (38) represent the bright and singular soliton solutions, respectively.

**4.2. Tanh—Coth method**

Assume $v = v(\xi)$, by using the ansatz [14]

$$Y = \tanh(\mu \xi), \quad (39)$$

that leads to the change of variables:

$$\frac{dv}{d\xi} = \mu(1 - Y^2) \frac{dv}{dY}, \quad (40)$$

$$\frac{d^2 v}{d\xi^2} = \mu^2 \left[-2Y(1 - Y^2) \frac{dv}{dY} + (1 - Y^2)^2 \frac{d^2 v}{dY^2}\right]. \quad (41)$$

For the next step, assume that the solution for Eq. (18) is expressed in the form

$$v(Y) = \sum_{i=0}^{p} a_i Y^i + \sum_{i=1}^{p} b_i Y^{-i}, \quad (42)$$

using the principle of the homogeneous balance method between the nonlinear term $v^3 v''''$ and the linear term $v^6$ from Eq. (18), then $3N + N + 4 = 6N$, which gives $N = 2$. Hence, Eq. (55) becomes

$$v(Y) = \left(a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}\right), \quad (43)$$

where $a_0,\ a_1, a_2, b_1$, and $b_2$ are constants to be determined. Then substitute Eq. (42) with their derivatives into Eq. (18), and for simplicity assume $a_0 = a_1 = b_1 = 0$ and $a_2 = b_2$ we obtain an overdetermined system

$$8 \mu^4 c_1 \sigma_1 [49 n^3 + 128 n^2 + 38 n + 2] + 4 \mu^2 n^3 c_1 \sigma_4 a_2^2$$

$$+ n^5 \beta_4 a_2^2 = 0,$$

$$8 \mu^2 c_1 \sigma_1 [56 n^3 + 62 n^2 - 2n - 8] + n^3 c_1 \sigma_4 a_2^2 = 0,$$

$$8 \mu^4 c_1 \sigma_1 [-83 n^3 - 106 n^2 + 26 n + 4] + 4 \mu^2 n^3 \beta_1$$

$$+ n^5 \beta_2 - 24 n^3 c_1 \sigma_4 a_2^2 + 6 n^5 \beta_4 a_2^2 = 0,$$

$$\mu^2 c_1 \sigma_1 [190 n^3 + 1131 n^2 - 601 n + 96] - n^3 \beta_1$$

$$+ 40 n^3 c_1 \sigma_4 a_2^2 = 0,$$

$$\mu^4 c_1 \sigma_1 [180 n^3 - 2632 n^2 + 1512 n + 272] - 32 \mu^2 n^3 \beta_1$$

$$+ 4 n^5 \beta_2 - 132 \mu^2 n^3 c_1 \sigma_4 a_2^2 + 15 n^5 \beta_4 a_2^2 = 0,$$

$$\mu^2 c_1 \sigma_1 [-1024 n^3 + 928 n^2 - 608 n + 128] + 48 n^3 \beta_1$$

$$+ 160 n^3 c_1 \sigma_4 a_2^2 = 0,$$

$$10 \mu^4 c_1 \sigma_1 [-301 n^3 + 458 n^2 - 218 n + 28]$$

$$-48 n^3 \mu^2 \beta_1 + 6 n^5 \beta_2 + 20 n^5 \beta_4 a_2^2$$

$$-184 n^3 \mu^2 c_1 \sigma_4 a_2^2 = 0. \quad (44)$$

Solving SAE (44) gives:

$$\mu = \frac{n}{2}\sqrt{\frac{-(a+6k^2c_1\sigma_1)n}{2(332+3n-2333n^2-2220n^3)c_1\sigma_1}}. \quad (45)$$

**Family I:**

$$a_2 = b_2 = \mp\frac{2\mu^2}{n}\sqrt{-\frac{2c_1\sigma_1[49n^3+128n^2+38n+2]}{n(4\mu^2 n c_1\sigma_4+n^2\beta_4)}}, \quad (46)$$

$$\Phi_1(x,t) = \left[\mp\frac{2\mu^2}{n}\sqrt{-\frac{2c_1\sigma_1[49n^3+128n^2+38n+2]}{n(4\mu^2 n c_1\sigma_4+n^2\beta_4)}} \times \{tanh^2(\mu(x-\gamma t))+coth^2(\mu(x-\gamma t))\}\right]^{\frac{1}{n}}$$

$$\times \exp[i(-\kappa x+\omega t+\theta_0). \quad (47)$$

**Family II:**

$$a_2 = b_2 = \mp\frac{2\mu}{n}\sqrt{\frac{2[-56n^3-62n^2+2n+8]}{n}}, \quad (48)$$

$$\Phi_2(x,t) = \left[\mp\frac{2\mu}{n}\sqrt{\frac{2[-56n^3-62n^2+2n+8]}{n}} \times \{tanh^2(\mu(x-\gamma t))+coth^2(\mu(x-\gamma t))\}\right]^{\frac{1}{n}}$$

$$\times \exp[i(-\kappa x+\omega t+\theta_0). \quad (49)$$

**Family III:**

$$a_2 = b_2 = \mp\frac{2\mu^2}{n}\sqrt{\frac{c_1\sigma_1[-83n^3-106n^2+26n+4]+4\mu^2 n^3\beta_1+n^5\beta_2}{3n(4c_1\sigma_4-n^3\beta_4)}}, \quad (50)$$

$$\Phi_3(x,t) = \left[\mp\frac{2\mu^2}{n}\sqrt{\frac{c_1\sigma_1[-83n^3-106n^2+26n+4]+4\mu^2 n^3\beta_1+n^5\beta_2}{3n(4c_1\sigma_4-n^3\beta_4)}} \times \{tanh^2(\mu(x-\gamma t))+coth^2(\mu(x-\gamma t))\}\right]^{\frac{1}{n}}$$

$$\times \exp[i(-\kappa x+\omega t+\theta_0). \quad (51)$$

**Family IV:**

$$a_2 = b_2 = \mp\frac{1}{2n}\sqrt{\frac{n^3\beta_1-\mu^2 c_1\sigma_1[190n^3+1131n^2-601n+96]}{10nc_1\sigma_4}}, \quad (52)$$

$$\Phi_4(x,t) = \left[\mp\frac{1}{2n}\sqrt{\frac{n^3\beta_1-\mu^2 c_1\sigma_1[190n^3+1131n^2-601n+96]}{10nc_1\sigma_4}} \times \{tanh^2(\mu(x-\gamma t))+coth^2(\mu(x-\gamma t))\}\right]^{\frac{1}{n}}$$

$$\times \exp[i(-\kappa x+\omega t+\theta_0). \quad (53)$$

**Family V:**

$$a_2 = b_2 = \mp\frac{1}{n}\sqrt{\frac{\mu^4 c_1\sigma_1\left[\begin{matrix}180n^3-2632n^2\\+1512n+272\end{matrix}\right]-32\mu^2 n^3\beta_1+4n^5\beta_2}{n(132\mu^2 c_1\sigma_4-15n^2\beta_4)}}, \quad (54)$$

$$\Phi_5(x,t) = \left[\mp\frac{1}{n}\sqrt{\frac{\mu^4 c_1\sigma_1[180n^3-2632n^2+1512n+272]-32\mu^2 n^3\beta_1+4n^5\beta_2}{n(132\mu^2 c_1\sigma_4-15n^2\beta_4)}} \times \{tanh^2(\mu(x-\gamma t))+coth^2(\mu(x-\gamma t))\}\right]^{\frac{1}{n}}$$

$$\times \exp[i(-\kappa x+\omega t+\theta_0). \quad (55)$$

**Family VI:**

$$a_2 = b_2 = \mp\frac{1}{4n}\sqrt{\frac{\mu^2 c_1\sigma_1[-1024n^3+928n^2-608n+128]+48n^3\beta_1}{-10nc_1\sigma_4}}, \quad (56)$$

$$\Phi_6(x,t) = \left[\mp\frac{1}{4n}\sqrt{\frac{\mu^2 c_1\sigma_1[-1024n^3+928n^2-608n+128]+48n^3\beta_1}{-10nc_1\sigma_4}} \times \{tanh^2(\mu(x-\gamma t))+coth^2(\mu(x-\gamma t))\}\right]^{\frac{1}{n}}$$

$$\times \exp[i(-\kappa x+\omega t+\theta_0). \quad (57)$$

**Family VII:**

$$a_2 = b_2 = \frac{1}{2n}\sqrt{\frac{10\mu^4 c_1\sigma_1[-301n^3+458n^2-218n+28]-48n^3\mu^2\beta_1+6n^5\beta_2}{n(46\mu^2 c_1\sigma_4-5n^2\beta_4)}}, \quad (58)$$

$$\Phi_7(x,t) = \left[\mp\frac{1}{2n}\sqrt{\frac{10\mu^4 c_1\sigma_1[-301n^3+458n^2-218n+28]-48n^3\mu^2\beta_1+6n^5\beta_2}{n(46\mu^2 c_1\sigma_4-5n^2\beta_4)}} \times \{tanh^2(\mu(x-\gamma t))+coth^2(\mu(x-\gamma t))\}\right]^{\frac{1}{n}}$$

$$\times \exp[i(-\kappa x+\omega t+\theta_0). \quad (59)$$

Figs. 1-3 display the numerical simulations for the bright soliton solution (37), conducted under specific conditions: $c_1 = 1$, $\sigma_1 = 1$, $\beta_4 = -1$, $\eta_2 = 1$, $\gamma = 1$, $\eta_1 = 1$, $p = 1$ and $q = 1$.

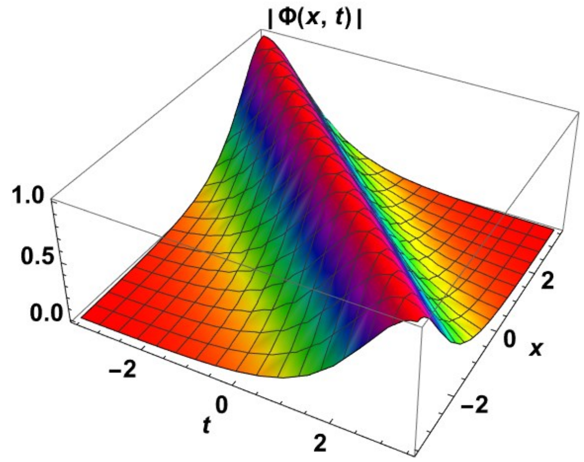


*Fig. 1. Surface plot of a bright soliton solution (color online)*

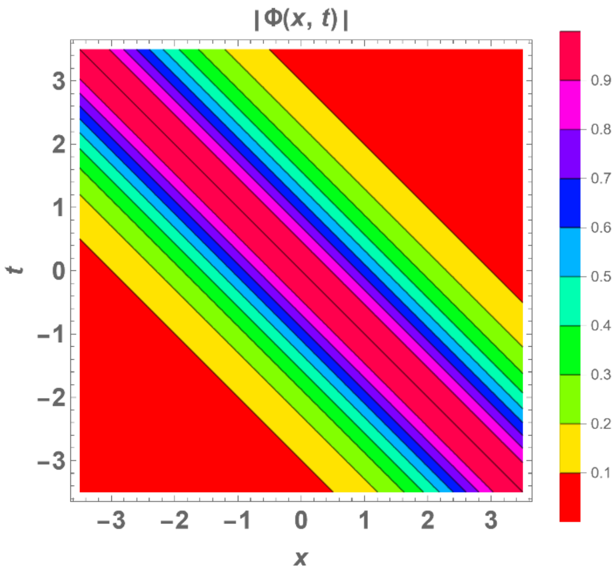


*Fig. 2. Contour plot of a bright soliton solution (color online)*

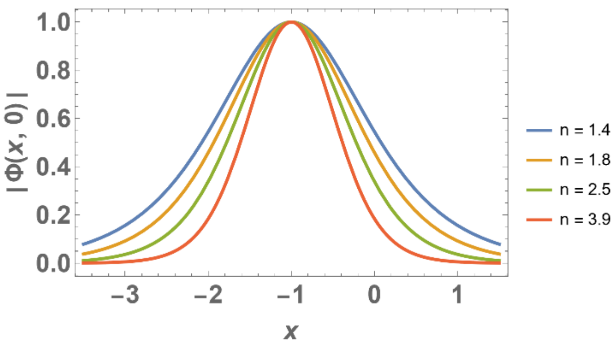


*Fig. 3. 2D plot of a bright soliton solution varying values of n (color online)*

## 5. Conclusions

The focus of this work is centered around the retrieval of optical solitons within the context of a concatenation model that incorporates power-law self-phase modulation. In this particular model, the behavior of optical solitons is examined with the assistance of Sardar's subequation approach, which is a mathematical method used to analyze nonlinear partial differential equations, and its variation. The tanh-coth approach, which utilizes hyperbolic tangent and hyperbolic cotangent functions, is employed to uncover solutions for dark-singular straddled optical solitons within the model. These solutions represent specific configurations of optical solitons that exhibit dark and singular characteristics. Furthermore, the work delves into establishing parameter constraints that dictate the existence of such solitons within the system. These constraints are crucial as they define the range of parameter values under which the solitons can manifest. Additionally, the study presents numerical simulations to complement the theoretical findings. These simulations involve computational methods to simulate the behavior of optical solitons under varying conditions, providing insights into their dynamics and stability.

## Disclosure

The authors claim there is no conflict of interest.

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