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A study on the conformable time-fractional Klein–Gordon equations with quadratic and cubic nonlinearities

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The nonlinear time-fractional Klein–Gordon equations are a class of fractional partial differential equations which are used for delineation of some physical phenomena in solid state physics, nonlinear optics, and quantum field theory. In this paper, the time-fractional Klein–Gordon equations with quadratic and cubic nonlinearities in the context of the conformable fractional derivative are explored via a recently developed approach named the $\exp(-\phi(\varepsilon))$-expansion method. Various families of solutions, such as the hyperbolic and trigonometric function solutions are formally achieved. Results reveal that the $\exp(-\phi(\varepsilon))$-expansion method is an efficient tool to derive the exact solutions of nonlinear fractional differential equations.

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1. Introduction

Nowadays, with the advent of computers and the computational convenience that they have provided, seeking the exact solutions of nonlinear fractional differential equations has gained considerable attention among mathematicians. Nonlinear fractional partial differential equations commonly appear in the problems in which many parameters are involved through their mathematical modelling. The nonlinear time-fractional Klein–Gordon equations are a class of such equations which have wide range of applications in several scientific disciplines, such as solid state physics, nonlinear optics, and quantum field theory [1]. There are different ways to establish the exact solutions of a nonlinear fractional partial differential equation; but a straightforward way is introducing a transformation which converts the original equation into a nonlinear ordinary differential equation and then solving the resulting equation. In this respect, a variety of robust methods, such as functional variable method [2-4], ansatz method [5-7], exp-function method [8-11], Kudryashov method [12,13], sub-equation method [14,15], and (G'/G)-expansion method [16,17] have been proposed. One of the new methods which benefits a wave transformation to generate the exact solutions of nonlinear fractional differential equations is the $\exp(-\phi(\varepsilon))$ -expansion method. The main idea of this algorithm is that the traveling wave solutions of equations can be expressed as a polynomial in terms of $\exp(-\phi(\varepsilon))$, in which $\phi(\varepsilon)$ satisfies a first-order ordinary differential equation. Here, some applications of this method are reviewed. Akbulut et al. [18] employed the $\exp(-\phi(\varepsilon))$-expansion method to establish several new exact solutions of the Zakharov Kuznetsov–Benjamin Bona Mahony and ill-posed Boussinesq equations. Kaplan and Bekir [19] applied the $\exp(-\phi(\varepsilon))$-expansion method to construct the exact solutions of the space-time fractional Jimbo–Miwa equation, and Koçak et al. [20] utilized the $\exp(-\phi(\varepsilon))$-expansion method to extract the complex function, hyperbolic function, and rational function solutions of the new coupled Konno–Oono equation. For more works, see [21-57]. In this paper, the $\exp(-\phi(\varepsilon))$-expansion method is employed to find the hyperbolic and trigonometric function solutions of the nonlinear Klein–Gordon equations with quadratic and cubic nonlinearities in the context of the conformable fractional derivative.

2. Preliminaries of the conformable fractional derivative

Lately, the conformable fractional derivative was established by Khalil et al. [58], which can resolve the deficiencies of the previous definitions. Here, the basic ideas of the conformable fractional derivative are mentioned [58-60].

Definition 1. Suppose $f:(0,\infty)\to R$ be a function. Then, the conformable fractional derivative of f of order α is defined as

$$T_\alpha(f)(t)=\lim_{\tau\to 0}\frac{f\left(t+\tau t^{1-\alpha}\right)-f(t)}{\tau},$$

for all $t>0$ and $\alpha\in(0,1]$.

Theorem 1. Suppose $\alpha\in(0,1]$, and f and g be α-differentiable at $t>0$. Then

$$T_\alpha(af+bg)=aT_\alpha(f)+bT_\alpha(g),\quad \forall\, a,b\in R.$$

$$T_\alpha\left(t^\mu\right)=\mu t^{\mu-\alpha},\quad \forall\, \mu\in R.$$

$$T_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}(t).$$

Theorem 2. Suppose $f:(0,\infty)\to R$ be a function such that f is differentiable and also α-differentiable. Let g be a function defined in the range of f and also differentiable. Then

$$T_\alpha(fog)(t)=t^{1-\alpha}g'(t)f'(g(t)).$$

3. Fundamental of the $\exp(-\phi(\varepsilon))$-expansion method

Consider a nonlinear partial differential equation with the conformable time-fractional derivative as

$$F\left(u,\frac{\partial^\alpha u}{\partial t^\alpha},\frac{\partial u}{\partial x},\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}},\frac{\partial^2 u}{\partial x^2},\ldots\right)=0. \tag{1}$$

Under the wave transformation $u(x,t)=f(\varepsilon)$ where $\varepsilon=x-l\left(t^\alpha/\alpha\right)$, Eq. (1) varies into a nonlinear ordinary differential equation as

$$G(f,f',f'',\ \ )=0. \tag{2}$$

Suppose that the solution of Eq. (2) can be considered as a truncated series in the following form

$$f(\varepsilon)=\sum_{n=0}^{N}a_n\left(\exp(-\phi(\varepsilon))\right)^n, \tag{3}$$

where a_n, $n=0,1,\ldots,N$ ($a_N\neq 0$) are constants to be evaluated later and $\phi(\varepsilon)$ is a function that satisfies a first-order auxiliary nonlinear ordinary equation as

$$\phi'(\varepsilon)=\exp(-\phi(\varepsilon))+\mu\exp(\phi(\varepsilon))+\lambda.$$

Now, there are several cases.

In the first case, when $\lambda^2-4\mu>0$ and $\mu\neq 0$, then

$$\phi_1(\varepsilon)=\ln\left(\frac{-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\varepsilon+C)\right)-\lambda}{2\mu}\right).$$

In the second case, when $\lambda^2-4\mu>0$, $\mu=0$, and $\lambda\neq 0$, then

$$\phi_2(\varepsilon)=-\ln\left(\frac{\lambda}{\cosh(\lambda(\varepsilon+C))+\sinh(\lambda(\varepsilon+C))-1}\right).$$

In the third case, when $\lambda^2-4\mu<0$ and $\mu\neq 0$, then

$$\phi_3(\varepsilon)=\ln\left(\frac{\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(\varepsilon+C)\right)-\lambda}{2\mu}\right).$$

The value of N is calculated by balancing the highest order nonlinear term with the highest order derivative term in Eq. (2). Setting Eq. (3) and its required derivatives in Eq. (2), gives

$$P(\exp(-\phi(\varepsilon)))=0. \tag{4}$$

By equating the coefficient of each power of $\exp(-\phi(\varepsilon))$ in (4) to zero, we will attain a set of nonlinear equations whose solution yields the exact solutions for Eq. (1).

Note 1. For $N=1$, the first two derivatives of $f(\varepsilon)$ are listed in the following

$$f'(\varepsilon)=-a_1\left(\exp(-2\phi(\varepsilon))+\lambda\exp(-\phi(\varepsilon))+\mu\right),$$

$$f''(\varepsilon)=a_1\begin{pmatrix}2\exp(-3\phi(\varepsilon))\\ +3\lambda\exp(-2\phi(\varepsilon))\\ +\left(\lambda^2+2\mu\right)\exp(-\phi(\varepsilon))+\lambda\mu\end{pmatrix}. \tag{5}$$

Note 2. For $N=2$, the first two derivatives of $f(\varepsilon)$ are listed in the following

$$f'(\varepsilon) = -a_1(\exp(-2\phi(\varepsilon)) + \lambda\exp(-\phi(\varepsilon)) + \mu) - 2a_2\begin{pmatrix}\exp(-3\phi(\varepsilon)) + \lambda\exp(-2\phi(\varepsilon)) \\ + \mu\exp(-\phi(\varepsilon))\end{pmatrix},$$

$$f''(\varepsilon) = a_1\begin{pmatrix}2\exp(-3\phi(\varepsilon)) + 3\lambda\exp(-2\phi(\varepsilon)) \\ + (\lambda^2+2\mu)\exp(-\phi(\varepsilon)) + \lambda\mu\end{pmatrix} + a_2\begin{pmatrix}6\exp(-4\phi(\varepsilon)) + 10\lambda\exp(-3\phi(\varepsilon)) \\ + (4\lambda^2+8\mu)\exp(-2\phi(\varepsilon)) \\ + 6\lambda\mu\exp(-\phi(\varepsilon)) + 2\mu^2\end{pmatrix}. \quad (6)$$

4. Application

In the current section, the $\exp(-\phi(\varepsilon))$-expansion method is exerted to find the hyperbolic and trigonometric function solutions of the nonlinear Klein–Gordon equations with quadratic and cubic nonlinearities in the context of the conformable fractional derivative. The computations are performed using the Maple package.

4.1. Conformable time-fractional Klein–Gordon equation with quadratic nonlinearity

Let's consider a time-fractional Klein–Gordon equation with quadratic nonlinearity in the sense of the conformable fractional derivative [39]

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} + a\frac{\partial^2 u(x,t)}{\partial x^2} + bu(x,t) + cu^2(x,t) = 0, \quad (7)$$

where $t>0$ and $0<\alpha\le 1$.
Under the transformation $u(x,t) = f(\varepsilon)$ in which $\varepsilon = x - l(t^\alpha/\alpha)$, Eq. (7) varies into an ordinary differential equation as

$$(l^2+a)f'' + bf + cf^2 = 0. \quad (8)$$

4.1.1. Implementation of the $\exp(-\phi(\varepsilon))$-expansion method

By balancing the second-order derivative of f and the nonlinear term f^2 in Eq. (8), we find $N=2$. So, Eq. (3) takes the form

$$f(\varepsilon) = a_0 + a_1\exp(-\phi(\varepsilon)) + a_2\exp(-2\phi(\varepsilon)). \quad (9)$$

By substituting (9) and (6) into Eq. (8) and equating the coefficient of each power of $\exp(-\phi(\varepsilon))$ to zero, we will get a set of nonlinear algebraic equations as

$$ba_0 + (a\lambda\mu + \lambda\mu l^2)a_1 + (2a\mu^2 + 2\mu^2 l^2)a_2 + ca_0^2 = 0,$$

$$(l^2\lambda^2 + a\lambda^2 + 2\mu l^2 + 2a\mu + b)a_1 + (6\lambda\mu l^2 + 6a\lambda\mu)a_2 + 2ca_0a_1 = 0,$$

$$(3\lambda l^2 + 3a\lambda)a_1 + (4l^2\lambda^2 + 4a\lambda^2 + 8\mu l^2 + 8a\mu + b)a_2 + ca_1^2 + 2ca_0a_2 = 0,$$

$$(2l^2+2a)a_1 + (10\lambda l^2 + 10a\lambda)a_2 + 2ca_1a_2 = 0,$$

$$(6l^2+6a)a_2 + ca_2^2 = 0.$$

Applying the Maple package, leads to

Case 1.

$$a_0 = -\frac{b(\lambda^2+2\mu)}{c(\lambda^2-4\mu)},\quad a_1 = -\frac{6b\lambda}{c(\lambda^2-4\mu)},$$
$$a_2 = -\frac{6b}{c(\lambda^2-4\mu)},\quad l = \frac{\sqrt{-(\lambda^2-4\mu)(a\lambda^2-4a\mu-b)}}{\lambda^2-4\mu}.$$

Consequently, the exact solutions of the conformable time-fractional Klein–Gordon equation with quadratic nonlinearity can be identified as

For $\lambda^2-4\mu>0$ and $\mu\ne 0$

$$u_{1,2}(x,t) = -\frac{b(\lambda^2+2\mu)}{c(\lambda^2-4\mu)} + \frac{12b\lambda\mu}{c(\lambda^2-4\mu)\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\times\left(x\pm\frac{\sqrt{-(\lambda^2-4\mu)\times(a\lambda^2-4a\mu-b)}}{\lambda^2-4\mu}\frac{t^\alpha}{\alpha}+C\right)\right)+\lambda\right)} - \frac{24b\mu^2}{c(\lambda^2-4\mu)\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\times\left(x\pm\frac{\sqrt{-(\lambda^2-4\mu)\times(a\lambda^2-4a\mu-b)}}{\lambda^2-4\mu}\frac{t^\alpha}{\alpha}+C\right)\right)+\lambda\right)^2}.$$

For $\lambda^2-4\mu>0$, $\mu=0$ and $\lambda\ne 0$

$$u_{3,4}(x,t)=-\frac{b}{c}-\frac{6b}{c\left(\cosh\left(\lambda\left(x\pm\frac{\sqrt{-\lambda^2(a\lambda^2-b)}}{\lambda^2}\frac{t^\alpha}{\alpha}+C\right)\right)+\sinh\left(\lambda\left(x\pm\frac{\sqrt{-\lambda^2(a\lambda^2-b)}}{\lambda^2}\frac{t^\alpha}{\alpha}+C\right)\right)-1\right)}-\frac{6b}{c\left(\cosh\left(\lambda\left(x\pm\frac{\sqrt{-\lambda^2(a\lambda^2-b)}}{\lambda^2}\frac{t^\alpha}{\alpha}+C\right)\right)+\sinh\left(\lambda\left(x\pm\frac{\sqrt{-\lambda^2(a\lambda^2-b)}}{\lambda^2}\frac{t^\alpha}{\alpha}+C\right)\right)-1\right)^2}.$$

For $\lambda^2-4\mu<0$ and $\mu\neq 0$

$$u_{5,6}(x,t)=-\frac{b(\lambda^2+2\mu)}{c(\lambda^2-4\mu)}+\frac{12b\lambda\mu}{c(\lambda^2-4\mu)\left(\sqrt{4\mu-\lambda^2}\times\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\left(x\pm\frac{\sqrt{-(\lambda^2-4\mu)(a\lambda^2-4a\mu-b)}}{\lambda^2-4\mu}\frac{t^\alpha}{\alpha}+C\right)\right)+\lambda\right)}-\frac{24b\mu^2}{c(\lambda^2-4\mu)\left(\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\times\left(x\pm\frac{\sqrt{-(\lambda^2-4\mu)(a\lambda^2-4a\mu-b)}}{\lambda^2-4\mu}\frac{t^\alpha}{\alpha}+C\right)\right)+\lambda\right)^2}.$$

Case 2.

$$a_0=\frac{6b\mu}{c(\lambda^2-4\mu)},\quad a_1=\frac{6b\lambda}{c(\lambda^2-4\mu)},$$
$$a_2=\frac{6b}{c(\lambda^2-4\mu)},\quad l=\frac{\sqrt{-(\lambda^2-4\mu)(a\lambda^2-4a\mu+b)}}{\lambda^2-4\mu}.$$

Hence, the exact solutions of the conformable time-fractional Klein–Gordon equation with quadratic nonlinearity can be listed as

For $\lambda^2-4\mu>0$ and $\mu\neq 0$

$$u_{7,8}(x,t)=\frac{6b\mu}{c(\lambda^2-4\mu)}-\frac{12b\lambda\mu}{c(\lambda^2-4\mu)\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\times\left(x\pm\frac{\sqrt{-(\lambda^2-4\mu)(a\lambda^2-4a\mu+b)}}{\lambda^2-4\mu}\frac{t^\alpha}{\alpha}+C\right)\right)+\lambda\right)}+\frac{24b\mu^2}{c(\lambda^2-4\mu)\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\times\left(x\pm\frac{\sqrt{-(\lambda^2-4\mu)(a\lambda^2-4a\mu+b)}}{\lambda^2-4\mu}\frac{t^\alpha}{\alpha}+C\right)\right)+\lambda\right)^2}.$$

For $\lambda^2-4\mu>0$, $\mu=0$ and $\lambda\neq 0$

$$u_{9,10}(x,t)=\frac{6b}{c\left(\cosh\left(\lambda\left(x\pm\frac{\sqrt{-\lambda^2(a\lambda^2+b)}}{\lambda^2}\frac{t^\alpha}{\alpha}+C\right)\right)+\sinh\left(\lambda\left(x\pm\frac{\sqrt{-\lambda^2(a\lambda^2+b)}}{\lambda^2}\frac{t^\alpha}{\alpha}+C\right)\right)-1\right)}+\frac{6b}{c\left(\cosh\left(\lambda\left(x\pm\frac{\sqrt{-\lambda^2(a\lambda^2+b)}}{\lambda^2}\frac{t^\alpha}{\alpha}+C\right)\right)+\sinh\left(\lambda\left(x\pm\frac{\sqrt{-\lambda^2(a\lambda^2+b)}}{\lambda^2}\frac{t^\alpha}{\alpha}+C\right)\right)-1\right)^2}.$$

For $\lambda^2-4\mu<0$ and $\mu\neq 0$

$$u_{11,12}(x,t)=\frac{6b\mu}{c(\lambda^2-4\mu)}-\frac{12b\lambda\mu}{c(\lambda^2-4\mu)\left(\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\times\left(x\pm\frac{\sqrt{-(\lambda^2-4\mu)(a\lambda^2-4a\mu+b)}}{\lambda^2-4\mu}\frac{t^\alpha}{\alpha}+C\right)\right)+\lambda\right)}+\frac{24b\mu^2}{c(\lambda^2-4\mu)\left(\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\times\left(x\pm\frac{\sqrt{-(\lambda^2-4\mu)(a\lambda^2-4a\mu+b)}}{\lambda^2-4\mu}\frac{t^\alpha}{\alpha}+C\right)\right)+\lambda\right)^2}.$$

4.2. Conformable time-fractional Klein–Gordon equation with cubic nonlinearity

Let us consider a time-fractional Klein–Gordon equation with cubic nonlinearity in the sense of the conformable fractional derivative [39]

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}+a\frac{\partial^2 u(x,t)}{\partial x^2}+bu(x,t)+cu^3(x,t)=0, \quad (10)$$

where $t>0$ and $0<\alpha\leq 1$.
Under the transformation $u(x,t)=f(\varepsilon)$ in which $\varepsilon=x-l\left(t^\alpha/\alpha\right)$, Eq. (10) turns into an ordinary differential equation as follows

$$\left(l^2+a\right)f''+bf+cf^3=0. \quad (11)$$

4.2.1. Implementation of the $\exp(-\phi(\varepsilon))$-expansion method

By balancing the second-order derivative of f and the nonlinear term f^3 in Eq. (11), we get $N=1$. Consequently, Eq. (3) becomes

$$f(\varepsilon)=a_0+a_1\exp(-\phi(\varepsilon)). \quad (12)$$

By substituting (12) and (5) into Eq. (11) and equating the coefficient of each power of $\exp(-\phi(\varepsilon))$ to zero, a set of nonlinear equations will be derived as

$$ba_0+\left(a\lambda\mu+\lambda\mu l^2\right)a_1+ca_0^3=0,$$
$$\left(l^2\lambda^2+a\lambda^2+2\mu l^2+2a\mu+b\right)a_1+3ca_0^2a_1=0,$$
$$\left(3\lambda l^2+3a\lambda\right)a_1+3ca_0a_1^2=0,$$
$$\left(2l^2+2a\right)a_1+ca_1^3=0.$$

Applying the Maple package, yields
Case 1.

$$a_0=-\frac{b\lambda}{\sqrt{-bc\left(\lambda^2-4\mu\right)}},\quad a_1=\frac{2\sqrt{-bc\left(\lambda^2-4\mu\right)}}{c\left(\lambda^2-4\mu\right)},$$
$$l=\frac{\sqrt{-\left(\lambda^2-4\mu\right)\left(a\lambda^2-4a\mu-2b\right)}}{\lambda^2-4\mu}.$$

Consequently, the exact solutions of the conformable time-fractional Klein–Gordon equation with cubic nonlinearity can be listed as

For $\lambda^2-4\mu>0$ and $\mu\neq 0$

$$u_{1,2}(x,t)=-\frac{b\lambda}{\sqrt{-bc\left(\lambda^2-4\mu\right)}}-\frac{4\mu\sqrt{-bc\left(\lambda^2-4\mu\right)}}{c\left(\lambda^2-4\mu\right)\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\times\left(x\pm\frac{\sqrt{-\left(\lambda^2-4\mu\right)\times\left(a\lambda^2-4a\mu-2b\right)}}{\lambda^2-4\mu}\frac{t^\alpha}{\alpha}+C\right)\right)+\lambda\right)}.$$

For $\lambda^2-4\mu>0$ and $\mu=0$

$$u_{3,4}(x,t)=-\frac{b\lambda}{\sqrt{-bc\lambda^2}}+\frac{2\sqrt{-bc\lambda^2}}{\lambda c\left(\cosh\left(\lambda\left(x\pm\frac{\sqrt{-\lambda^2\left(a\lambda^2-2b\right)}}{\lambda^2}\frac{t^\alpha}{\alpha}+C\right)\right)+\sinh\left(\lambda\left(x\pm\frac{\sqrt{-\lambda^2\left(a\lambda^2-2b\right)}}{\lambda^2}\frac{t^\alpha}{\alpha}+C\right)\right)-1\right)}.$$

For $\lambda^2-4\mu<0$ and $\mu\neq 0$

$$u_{5,6}(x,t)=-\frac{b\lambda}{\sqrt{-bc\left(\lambda^2-4\mu\right)}}-\frac{4\mu\sqrt{-bc\left(\lambda^2-4\mu\right)}}{c\left(\lambda^2-4\mu\right)\left(\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\times\left(x\pm\frac{\sqrt{-\left(\lambda^2-4\mu\right)\times\left(a\lambda^2-4a\mu-2b\right)}}{\lambda^2-4\mu}\frac{t^\alpha}{\alpha}+C\right)\right)+\lambda\right)}.$$

Case 2.

$$a_0=\frac{b\lambda}{\sqrt{-bc\left(\lambda^2-4\mu\right)}},\quad a_1=-\frac{2\sqrt{-bc\left(\lambda^2-4\mu\right)}}{c\left(\lambda^2-4\mu\right)},$$
$$l=\frac{\sqrt{-\left(\lambda^2-4\mu\right)\left(a\lambda^2-4a\mu-2b\right)}}{\lambda^2-4\mu}.$$

Hence, the exact solutions of the conformable time-fractional Klein–Gordon equation with cubic nonlinearity can be identified as

For $\lambda^2-4\mu>0$ and $\mu\neq 0$

$$u_{7,8}(x,t)=\frac{b\lambda}{\sqrt{-bc(\lambda^2-4\mu)}}+\frac{4\mu\sqrt{-bc(\lambda^2-4\mu)}}{c(\lambda^2-4\mu)\left(\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\times\left(x\pm\frac{\sqrt{-(\lambda^2-4\mu)\times(a\lambda^2-4a\mu-2b)}}{\lambda^2-4\mu}\frac{t^\alpha}{\alpha}+C\right)\right)+\lambda\right)}.$$

For $\lambda^2-4\mu>0$ and $\mu=0$

$$u_{9,10}(x,t)=\frac{b\lambda}{\sqrt{-bc\lambda^2}}-\frac{2\sqrt{-bc\lambda^2}}{\lambda c\left(\cosh\left(\lambda\left(x\pm\frac{\sqrt{-\lambda^2(a\lambda^2-2b)}}{\lambda^2}\frac{t^\alpha}{\alpha}+C\right)\right)+\sinh\left(\lambda\left(x\pm\frac{\sqrt{-\lambda^2(a\lambda^2-2b)}}{\lambda^2}\frac{t^\alpha}{\alpha}+C\right)\right)-1\right)}.$$

For $\lambda^2-4\mu<0$ and $\mu\neq 0$

$$u_{11,12}(x,t)=\frac{b\lambda}{\sqrt{-bc(\lambda^2-4\mu)}}+\frac{4\mu\sqrt{-bc(\lambda^2-4\mu)}}{c(\lambda^2-4\mu)\left(\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\times\left(x\pm\frac{\sqrt{-(\lambda^2-4\mu)\times(a\lambda^2-4a\mu-2b)}}{\lambda^2-4\mu}\frac{t^\alpha}{\alpha}+C\right)\right)+\lambda\right)}.$$

5. Conclusion

In this paper, a novel technique called the $\exp(-\phi(\varepsilon))$-expansion method along with the fractional transform was successfully exerted to solve the time-fractional Klein–Gordon equations with quadratic and cubic nonlinearities in the context of the conformable fractional derivative. As a result, various families of solutions, such as the hyperbolic and trigonometric function solutions were formally achieved. The technique is actually effective; such that it can be used to a wide diversity of nonlinear fractional differential equations in mathematical physics.

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