OPTOELECTRONICS AND ADVANCED MATERIALS – RAPID COMMUNICATIONS Vol. 9, No. 11-12, Nov. – Dec. 2015, p. 1342 - 1346

Bifurcation analysis and bright soliton of generalized resonant dispersive nonlinear Schrödinger's equation

MOHAMMAD MIRZAZADEH[a], QIN ZHOU[b], ESSAID ZERRAD[c], MOHAMMAD F. MAHMOOD[d], ANJAN BISWAS[e,f,*], MILIVOJ BELIC[g]
[a]*Department of Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran*
[b]*School of Electronics and Information Engineering, Wuhan Donghu University, Wuhan, 430212, P.R. China*
[c]*Department of Physics and Engineering, Delaware State University, Dover, DE 19901-2277, USA*
[d]*Department of Mathematics, Howard University, Washington, DC-20059, USA*
[e]*Department of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA*
[f]*Department of Mathematics, Faculty of Science, King Abdulaziz University Jeddah-21589, Saudi Arabia*
[g]*Science Program, Texas A & M University at Qatar, PO Box 23874, Doha, Qatar*

This paper carries out bifurcation analysis of the generalized resonant dispersive nonlinear Schrödinger's equation. This lead to the retrieval of bright 1-soliton solution to the model equation along with singular periodic solutions. There are constraint conditions in place that guarantees existence of the soliton.

(Received March 26, 2015; accepted October 28, 2015)

Keywords: Solitons, Integrability, Bifurcation

1. Introduction

Optical solitons is a growing and fascinating area of research in the area of Physics and Engineering. The dynamics of soliton molecules has the marvelous capacity of carrying loads of information across the globe in a matter of a few femtoseconds. The commonly studied model in this context is the nonlinear Schrödinger's equation (NLSE) [1-15]. There are several issues with soliton transmission, across trans-continental and trans-oceanic distances, based on this model. Therefore generalized model is proposed that is frequently studied as opposed to the regular NLSE. There are bends, kinks and other aspects that leads to the departure of the model from regular NLSE that compels the study of solitons with a model that carries generalized flavor [1, 2, 10-12]. This model is the generalized dispersive NLSE. To top it off, the model in this paper includes quantum or Bohm potential (also known as resonant term) that appears in the context of chiral solitons in quantum Hall effect which is also seen in the context of Madelung fluid in quantum mechanics. This model is thus being referred to as generalized resonant dispersive NLSE (GRD-NLSE). Bifurcation analysis will be carried out for this model that will lead to soliton solutions as well as singular periodic solutions.

2. Mathematical model

The model equation that will be studied in this paper is given by the GRD-NLSE and it is: [1, 2, 4, 5]

$$\begin{aligned} & i\left(|\psi|^{n-1}\psi\right)_t + a\left(|\psi|^{n-1}\psi\right)_{xx} \\ & + b|\psi|^m\psi + c\left\{\frac{(|\psi|^n)_{xx}}{|\psi|}\right\}\psi = 0 \end{aligned} \tag{1}$$

For this model, $\psi(x,t)$ is the wave profile and represents the complex valued function. The first term is the nonlinear temporal evolution, while a and b respectively represent the coefficients of the generalized group velocity dispersion (GVD) and power law nonlinearity. Then c represent the coefficient of the resonant term. The parameter m represents power law nonlinearity. When $m = 2$, this model equation condenses to Kerr law nonlinearity that is also referred to cubic NLSE. Finally, the parameter n dictates the generalized evolution and generalized GVD. For $n = 1$, this model collapses to the regular NLSE. This parameter n thus maintains the evolution and GVD on a generalized setting.

Bifurcation analysis for this model that will lead to the corresponding dynamical system with the possible fixed points. The integration of this model will subsequently follow from the analysis that will lead to a bright 1-soliton solution to GRD-NLSE. The constraint conditions will naturally emerge from the analysis, for the existence of the bright soliton solution.

3. Phase portraits and qualitative analysis

We assume that the traveling wave solutions of Eq. (1) is of the form [7, 8]

$$\psi(x,t)=U(\xi)e^{i\Phi(x,t)} \tag{2}$$

where $U(\xi)$ represents the shape of the pulse and

$$\xi=\alpha_0 t+\alpha_1 x \tag{3}$$

$$\Phi(x,t)=-\kappa x+\omega t \tag{4}$$

In Eq. (2), the function $\Phi(x,t)$ is the phase component of the soliton. Then, in Eq. (4), κ is the soliton frequency, while ω is the wave number of the soliton. Finally in Eq. (3), α_0 is the velocity of the soliton. By replacing Eq. (2) into Eq. (1) and separating the real and imaginary parts of the result, we have

$$\alpha_0=2\kappa a\alpha_1 \tag{5}$$

and

$$\begin{aligned}&\alpha_1^2(a+c)(U^n)''\\&-(\omega+a\kappa^2)U^n+bU^{m+1}=0\end{aligned} \tag{6}$$

Now, we use the transformation

$$U(\xi)=\varphi^{\frac{1}{n}}(\xi) \tag{7}$$

that will reduce Eq. (6) into the ODE

$$\begin{aligned}&\alpha_1^2(a+c)\varphi''\\&-(\omega+a\kappa^2)\varphi+b\varphi^{\frac{m+1}{n}}=0\end{aligned} \tag{8}$$

To facilitate discussions, we let

$$\theta=\frac{(\omega+a\kappa^2)}{\alpha_1^2(a+c)} \tag{9-1}$$

$$\delta=\frac{c}{\alpha_1^2(a+c)} \tag{9-2}$$

Letting $\varphi'=z$, then we get the following planar system:

$$\frac{d\varphi}{d\xi}=z \tag{10-1}$$

$$\frac{dz}{d\xi}=\theta\varphi-\delta\varphi^{\frac{m+1}{n}} \tag{10-2}$$

Obviously, the above system (10) is a Hamiltonian system with Hamiltonian function

$$H(\varphi,z)=z^2-\theta\varphi^2+\frac{2n\delta}{m+1+n}\varphi^{\frac{m+1+n}{n}} \tag{11}$$

In order to investigate the phase portrait of (10), set

$$f(\varphi)=\theta\varphi-\delta\varphi^{\frac{m+1}{n}} \tag{12}$$

Obviously, when $\theta\delta>0$, $f(\varphi)$ has three zero points, φ_-, φ_0 and φ_+, which are given as follows:

$$\varphi_-=-\left(\frac{\theta}{\delta}\right)^{\frac{n}{m+1-n}} \tag{13-1}$$

$$\varphi_0=0 \tag{13-2}$$

$$\varphi_+=\left(\frac{\theta}{\delta}\right)^{\frac{n}{m+1-n}} \tag{13-3}$$

When $\theta\delta\le 0$, $f(\varphi)$ has only one zero point

$$\varphi_0=0 \tag{14}$$

Letting $(\varphi_l,0)$ be one of the singular points of system (10), then the characteristic values of the linearized system of system (10) at the singular points $(\varphi_l,0)$ are

$$\lambda_\pm=\pm\sqrt{f'(\varphi_l)} \tag{15}$$

From the qualitative theory of dynamical systems, we know the following.

(I) If $f'(\varphi_l)>0$, $(\varphi_l,0)$ is a saddle point.

(II) If $f'(\varphi_l)<0$, $(\varphi_l,0)$ is a center point.

(III) If $f'(\varphi_l)=0$, $(\varphi_l,0)$ is a degenerate saddle point.

Therefore, we obtain the bifurcation phase portraits of system (10) in the figure.

$$H(\varphi, z) = h \tag{16}$$

where h is Hamiltonian.

Next, we consider the relations between the orbits of (10) and the Hamiltonian h. Set

$$h^* = |H(\varphi_+, 0)| = |H(\varphi_-, 0)| \tag{17}$$

According to Fig. 1, we get the following propositions [5,6,7].

Proposition 1. Suppose that $\theta > 0$ and $\delta > 0$, one has the following.

(I) When $h \le h^*$, system (10) does not have any closed orbits.

(II) When $-h^* < h < 0$, system (10) has two periodic orbits Γ_1 and Γ_2.

(III) When $h = 0$, system (10) has two homoclinic orbits Γ_3 and Γ_4.

(IV) When $h > 0$, system (10) has a periodic orbit Γ_5.

Proposition 2. Suppose that $\theta < 0$ and $\delta < 0$, one has the following.

(I) When $h < 0$ and $h > h^*$, system (10) does not have any closed orbits.

(II) When $0 < h < h^*$, system (10) has three periodic orbits Γ_6, Γ_7 and Γ_8.

(III) When $h = 0$, system (10) has two periodic orbits Γ_9 and Γ_{10}.

(IV) When $h = h^*$, system (10) has two heteroclinic orbits Γ_{11} and Γ_{12}.

Proposition 3. (I) When $\theta \ge 0$, $\delta > 0$, and $h > 0$, system (10) has a periodic orbits.

(II) When $\theta \le 0$, $\delta < 0$, system (10) does have not any closed orbits.

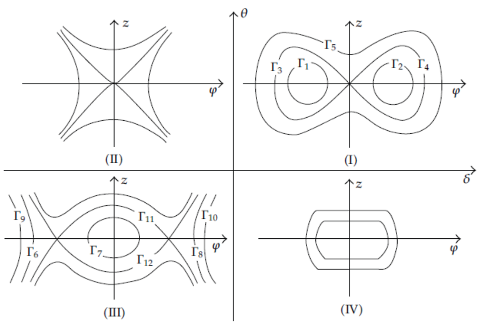

Fig. 1. The bifurcation phase portraits of system (11). (I) $\delta > 0$, $\theta > 0$, *(II)* $\delta < 0$, $\theta \ge 0$, *(III)* $\delta < 0$, $\theta < 0$, *(IV)* $\delta > 0$, $\theta \le 0$.

From the qualitative theory of dynamical systems, we know that a smooth solitary wave solution of a partial differential system corresponds to a smooth homoclinic orbit of a traveling wave equation. A smooth kink wave solution or a unbounded wave solution corresponds to a smooth heteroclinic orbit of a traveling wave equation. Similarly, a periodic orbit of a traveling wave equation corresponds to a periodic traveling wave solution of a partial differential system. According to the above analysis, we have the following propositions.

Proposition 4. If $\theta > 0$ and $\delta > 0$, one has the following.

(I) When $-h^* < h < 0$, Eq. (1) has two periodic wave solutions (corresponding to the periodic orbits Γ_1 and Γ_2 in Figure 1.)

(II) When $h = 0$, Eq. (1) has two solitary wave solutions (corresponding to the homoclinic orbits Γ_3 and Γ_4 in Fig. 1)

(III) When $h > 0$, Eq. (1) has two periodic wave solutions (corresponding to the periodic orbit Γ_5 in Figure 1.)

Proposition 5. If $\theta < 0$ and $\delta < 0$, one has the following.

(I) When $0 < h < h^*$, Eq. (1) has two periodic wave solutions (corresponding to the periodic orbit Γ_7 in Fig. 1) and two periodic blow-up wave solutions (corresponding to the periodic orbits Γ_6 and Γ_8 in Figure 1).

(II) When $h = 0$, Eq. (1) has periodic blow-up wave solutions (corresponding to the periodic orbits Γ_9 and Γ_{10} in Fig. 1).

(III) When $h = h^*$, Eq. (1) has two kink profile solitary wave solutions. (corresponding to the heteroclinic orbits Γ_{11} and Γ_{12} in Fig. 1).

4. Exact traveling wave solutions

Firstly, we will obtain the explicit expressions of traveling wave solutions for Eq. (1) when $\theta > 0$ and $\delta > 0$. From the phase portrait, we see that there are two symmetric homoclinic orbits Γ_3 and Γ_4 connected at the saddle point $(0,0)$. In (φ, z)-plane the expressions of the homoclinic orbits are given as

$$z = \pm\sqrt{\frac{2n\delta}{m+1+n}}\varphi \times \sqrt{-\varphi^{\frac{m+1-n}{n}} + \frac{(m+1+n)\theta}{2n\delta}} \tag{18}$$

Substituting (18) into $d\varphi/d\xi = z$ and integrating them along the orbits Γ_3 and Γ_4, we have

$$\pm\int_{\varphi_1}^{\varphi}\frac{1}{\sqrt{-s^{\frac{m+1-n}{n}}+\frac{(m+1+n)\theta}{2n\delta}}}ds = \sqrt{\frac{2n\delta}{m+1+n}}\int_0^{\xi}ds \tag{19-1}$$

$$\pm\int_{\varphi_2}^{\varphi}\frac{1}{\sqrt{-s^{\frac{m+1-n}{n}}+\frac{(m+1+n)\theta}{2n\delta}}}ds = \sqrt{\frac{2n\delta}{m+1+n}}\int_0^{\xi}ds \tag{19-2}$$

where $\varphi_1 = -\left(\frac{(m+1+n)\theta}{2n\delta}\right)^{\frac{n}{m+1-n}}$ and $\varphi_1 = \left(\frac{(m+1+n)\theta}{2n\delta}\right)^{\frac{n}{m+1-n}}$.

The above integrals give

$$\varphi = \left\{\sqrt{\frac{(m+1+n)\theta}{2n\delta}} \times \operatorname{sech}\left(\frac{m+1-n}{2n}\sqrt{\theta}\xi\right)\right\}^{\frac{2n}{m+1-n}} \tag{20-1}$$

$$\varphi = -\left\{\sqrt{\frac{(m+1+n)\theta}{2n\delta}} \times \operatorname{sech}\left(\frac{m+1-n}{2n}\sqrt{\theta}\xi\right)\right\}^{\frac{2n}{m+1-n}} \tag{20-2}$$

Noting (2), (5), (7) and (9), we get the following solitary wave solutions:

$$\psi(x,t) = \left[\sqrt{\frac{(m+1+n)(\omega+a\kappa^2)}{2nb}}\operatorname{sech}\left\{\frac{m+1-n}{2n}\times\sqrt{\frac{\omega+a\kappa^2}{\alpha_1^2(a+c)}}(2\kappa a\alpha_1 t+\alpha_1 x)\right\}\right]^{\frac{2}{m+1-n}} \tag{21}$$

Secondly, we will obtain the explicit expressions of traveling wave solutions for Eq. (1) when $\theta<0$ and $\theta>0$. From the phase portrait, we note that there are two special orbits Γ_9 and Γ_{10}, which have the same Hamiltonian with that of the center point $(0,0)$. In (φ, z)-plane the expressions of the orbits are given as

$$z = \pm\sqrt{-\frac{2n\delta}{m+1+n}}\varphi \times\sqrt{\varphi^{\frac{m+1-n}{n}}-\frac{(m+1+n)\theta}{2n\delta}} \tag{22}$$

Substituting (22) into $d\varphi/d\xi = z$ and integrating them along the orbits Γ_9 and Γ_{10}, we have

$$\pm\int_{\varphi}^{+\infty}\frac{1}{s\sqrt{s^{\frac{m+1-n}{n}}+\frac{(m+1+n)\theta}{2n\delta}}}ds = \sqrt{-\frac{2n\delta}{m+1+n}}\int_0^{\xi}ds \tag{23-1}$$

$$\pm\int_{\varphi_4}^{\varphi}\frac{1}{s\sqrt{s^{\frac{m+1-n}{n}}-\frac{(m+1+n)\theta}{2n\delta}}}ds = \sqrt{-\frac{2n\delta}{m+1+n}}\int_0^{\xi}ds \tag{23-2}$$

where $\varphi_4 = \left(\frac{(m+1+n)\theta}{2n\delta}\right)^{\frac{n}{m+1-n}}$.

Completing the above integrals we obtain

$$\varphi = \pm\left\{\sqrt{\frac{(m+1+n)\theta}{2n\delta}}\sec\left(\frac{m+1-n}{2n}\sqrt{-\theta}\xi\right)\right\}^{\frac{2n}{m+1-n}} \tag{24-1}$$

$$\varphi = \pm\left\{\sqrt{\frac{(m+1+n)\theta}{2n\delta}}\,\csc\left(\frac{m+1-n}{2n}\sqrt{-\theta}\,\xi\right)\right\}^{\frac{2n}{m+1-n}} \tag{24-2}$$

Noting (2), (5), (7) and (9), we get the following periodic blow-up wave solutions:

$$\psi(x,t) = \pm\left[\sqrt{\frac{(m+1+n)(\omega+a\kappa^2)}{2nb}}\,\sec\left\{\frac{m+1-n}{2n}\times\sqrt{-\frac{\omega+a\kappa^2}{\alpha_1^2(a+c)}}(2\kappa a\alpha_1 t+\alpha_1 x)\right\}\right]^{\frac{2}{m+1-n}} \tag{25}$$

and

$$\psi(x,t) = \pm\left[\sqrt{\frac{(m+1+n)(\omega+a\kappa^2)}{2nb}}\,\csc\left\{\frac{m+1-n}{2n}\times\sqrt{-\frac{\omega+a\kappa^2}{\alpha_1^2(a+c)}}(2\kappa a\alpha_1 t+\alpha_1 x)\right\}\right]^{\frac{2}{m+1-n}} \tag{26}$$

5. Conclusions

This paper secured bright 1-soliton solution to GRD-NLSE that stands as a generalized model to NLSE. Bifurcation analysis was carried out to secure this solution. There are constraint conditions that guarantee the existence of the soliton solution. The results of this paper stand on a very strong footing. Later, this equation will be extended to perturbation terms where these perturbations will also be studied with a generalized flavor. The results of those analysis will be revealed later. Additionally, the model will be studied with time-dependent coefficients that will lead to much improved results that is a much closer to realistic situation. This is just a tip of the iceberg.

Acknowledgments

This research is funded by Qatar National Research Fund (QNRF) under the grant number NPRP 6-021-1-005. The fifth and sixth authors (AB & MB) thankfully acknowledge this support from QNRF. The authors declare that there is no conflict of interest.

References

[1] A. Biswas, C. M. Khalique, Nonlinear Dynamics **63**, 623 (2011).
[2] A. Biswas, C. M. Khalique, Chinese Journal of Physics **51**, 157 (2013).
[3] Y. Geng, J. Li, Applied Mathematics and Computation **195**, 420 (2008).
[4] H. Triki, T. Hayat, O. M. Aldossary, A. Biswas, Optics and Laser Technology **44**, 2223 (2012).
[5] M. Eslami, M. Mirzazadeh, A. Biswas, Journal of Modern Optics **60**, 1627 (2013).
[6] M. Song, B. S. Ahmed, A. Biswas, Journal of Applied Mathematics **2013**, 972416 (2013).
[7] M. Song, Z. Liu, E. Zerrad, A. Biswas, Applied Mathematics and Information Sciences **7**, 1333 (2013).
[8] A. Biswas, M. Song, Communications in Nonlinear Science and Numerical Simulations **18**, 1676 (2013).
[9] Z. Yan, Physics Letters A **357**, 196 (2006).
[10] Z. Y. Zhang, Z. H. Liu, X. J. Miao, Y. Z. Chen, Applied Mathematics and Computation **216**, 3064 (2010).
[11] Z. Y. Zhang, Z. H. Liu, X. J. Miao, Y. Z. Chen, Physics Letters A **375**, 1275 (2011).
[12] Z. Y. Zhang, Y. L. Li, Z. H. Liu, X. J. Miao, Communications in Nonlinear Science and Numerical Simulation **16**, 3097 (2011).
[13] Q. Zhou, D. Yao, F. Chen, W. Li, Journal of Modern Optics **60**, 854 (2013).
[14] Q. Zhou, Journal of Modern Optics **61**, 500 (2014).
[15] Q. Zhou, Optik **125**, 3142 (2014).

*Corresponding author: biswas.anjan@gmail.com