Counting the number of dominating sets of cactus chains

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Let G be a simple graph of order n. The domination polynomial of G is the polynomial $D(G,x)=\sum_{i=\gamma(G)}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i and $\gamma(G)$ is the domination number of G. The number of dominating sets of a graph G is $D(G,1)$. In this paper we consider cactus chains with triangular and square blocks and study their domination polynomials.

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1. Introduction

Let $G=(V,E)$ be a simple graph. For any vertex $v\in V(G)$, the *open neighborhood* of v is the set $N(v)=\{u\in V(G)\,|\,\{u,v\}\in E(G)\}$ and the *closed neighborhood* of v is the set $N[v]=N(v)\cup\{v\}$. For a set $S\subseteq V(G)$, the open neighborhood of S is $N(S)=\bigcup_{v\in S}N(v)$ and the closed neighborhood of S is $N[S]=N(S)\cup S$. A set $S\subseteq V(G)$ is a *dominating set* if $N[S]=V$ or equivalently, every vertex in $V(G)\setminus S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. For a detailed treatment of these parameters, the reader is referred to [10]. Let $D(G,i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G,i)=|D(G,i)|$. The *domination polynomial* $D(G,x)$ of G is defined as $D(G,x)=\sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$, where $\gamma(G)$ is the domination number of G (see [2, 5]). Obviously, the number of dominating sets of a graph G is $D(G,1)$ (see [4, 13]). Recently the number of the dominating sets of graph G, i.e., $D(G,1)$ has been considered and studied in [18] with a different approach.

Domination theory have many applications in sciences and technology (see [10]). Recently the dominating set has found application in the assignment of structural domains in complex protein structures, which is an important task in bio-informatics ([8]).

We recall that the Hosoya index $Z(G)$ of a molecule graph G, is the number of matching sets, and the Merrifield-Simmons index $i(G)$ of graph G, is the number of independent sets. The Hosoya index of a graph has application to correlations with boiling points, entropies, calculated bond orders, as well as for coding of chemical structures. The Merrifield-Simmons index is one of the most popular topological indices in chemistry. For more information of these two indices see [1,15, 16, 19]. Note that $Z(G)$ and $i(G)$ can be study by the value of matching polynomial and independence polynomial at 1.

In this paper we consider a class of simple linear polymers called cactus chains. Cactus graphs were first known as Husimi trees; they appeared in the scientific literature some sixty years ago in papers by Husimi and Riddell concerned with cluster integrals in the theory of condensation in statistical mechanics [9,11,17]. We refer the reader to papers [7, 14] for some aspects of domination in cactus graphs.

A cactus graph is a connected graph in which no edge lies in more than one cycle. Consequently, each block of a cactus graph is either an edge or a cycle. If all blocks of a cactus G are cycles of the same size i, the cactus is i-uniform. A triangular cactus is a graph whose blocks are triangles, i.e., a 3-uniform cactus. A vertex shared by two or more triangles is called a cut-vertex. If each triangle of a triangular cactus G has at most two cut-vertices, and each cut-vertex is shared by exactly two triangles, we say that G is a chain triangular cactus. By replacing triangles in this definitions by cycles of length 4 we obtain cacti whose every block is C_4. We call such cacti square cacti. Note that the internal squares may differ in the way they connect to their neighbors. If their cut-vertices are adjacent, we say that such a square is an ortho-square; if the cut-vertices are not adjacent, we call the square a para-square.

In Section 2 we study the domination polynomial of the chain triangular cactus with two approach. In Section 3 we study the domination polynomials of chains of squares.

2. Domination polynomials of the chain triangular cactus

We call the number of triangles in G, the length of the chain. An example of a chain triangular cactus is shown in Fig. 1. Obviously, all chain triangular cacti of the same length are isomorphic. Hence, we denote the chain triangular cactus of length n by T_n. In this paper we investigate the domination polynomial of T_n by two different approach.

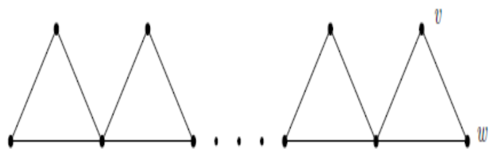

Fig. 1. The chain triangular cactus.

2.1 Computation of $D(T_n, x)$ using recurrence relation

In the first subsection, we use results and recurrence relations of the domination polynomial of a graph to find a recurrence relation for $D(T_n, x)$.

We need the following theorem:

Theorem 1. [5] *If a graph G consists of k components $G_1,\ldots,G_k$, then $D(G,x)=\prod_{i=1}^{k} D(G_i,x)$.*

The vertex contraction G/u of a graph G by a vertex u is the operation under which all vertices in $N(u)$ are joined to each other and then u is deleted (see[20]).

The following theorem is useful for finding the recurrence relations for the domination polynomials of arbitrary graphs.

Theorem 2. [3,12] *Let G be a graph. For any vertex u in G we have*

$$D(G,x)=xD(G/u,x)+D(G-u,x)+xD(G-N[u],x)-(1+x)p_u(G,x),$$

where $p_u(G,x)$ is the polynomial counting the dominating sets of $G-u$ which do not contain any vertex of $N(u)$ in G.

Domination polynomial satisfies a recurrence relation for arbitrary graphs which is based on the edge and vertex elimination operations. The recurrence uses composite operations, e.g. $G-e/u$, which stands for $(G-e)/u$.

Theorem 3 . [12] *Let G be a graph. For every edge $e=\{u,v\}\in E$,*

$$D(G,x)=D(G-e,x)+\frac{x}{x-1}[D(G-e/u,x)+D(G-e/v,x)$$
$$-D(G/u,x)-D(G/v,x)-D(G-N[u],x)-D(G-N[v],x)$$

We use for graphs $G=(V,E)$ the following vertex operation, which is commonly found in the literature. Let $v\in V$ be a vertex of G. A vertex appending $G+e$ (or $G+\{v,\cdot\}$) denotes the graph $(V\cup\{v'\},E\cup\{v,v'\})$ obtained from G by adding a new vertex v' and an edge $\{v,v'\}$ to G.

The following theorem gives recurrence relation for the domination polynomial of T_n.

Theorem 4 . *For every $n\geq 3$,*
$$D(T_n,x)=(x^2+2x)D(T_{n-1},x)+(x^2+x)D(T_{n-2},x),$$
with initial condition $D(T_1,x)=x^3+3x^2+3x$ and $D(T_2,x)=x^5+5x^4+10x^3+8x^2+x$.

Proof. Consider the graph T_n as shown in the following Fig. 1. Since T_n/u is isomorphic to T_n-u and $p_u(T_n,x)=0$, by Theorem 2 we have:

$$D(T_n,x)=xD(T_n/u,x)+D(T_n-u,x)+xD(T_n-N[u],x)$$
$$-(1+x)p_u(T_n,x)$$
$$=(x+1)D(T_{n-1}+e,x)+xD(T_{n-2}+e,x).$$

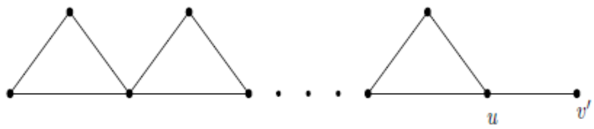

Fig. 2. The Graph $T_{n-1}+e$.

Note we use Theorems 1 and 2 to obtain the domination polynomial of the graph $T_{n-1}+e$ (see Fig. 2). Suppose that v' be a vertex of degree 1 in graph $T_{n-1}+e$ and let u be its neighbor. Note that in this case $p_u(T_{n-1}+e,x)=0$. We deduce that for each $n\in\mathbb{N}$, $D(T_{n-1}+e,x)=x[D(T_{n-1},x)+D(T_{n-2}+e,x)+D(T_{n-3}+e,x)]$. Therefore by equation (1) and this equality we have

$$D(T_n,x)=(x^2+x)(D(T_{n-1},x)+D(T_{n-3}+e,x))+(x^2+2x)D(T_{n-2}+e,x).$$

Now it's suffices to prove the following equality:

$$(x^2+x)D(T_{n-3}+e,x)+(x^2+2x)D(T_{n-2}+e,x)=xD(T_{n-1},x)+(x^2+x)D(T_{n-2},x).$$

For this purpose we use Theorem 2 for $D(T_{n-1},x)$. We have

$$xD(T_{n-1},x)=(x^2+x)D(T_{n-2}+e,x)+x^2D(T_{n-3}+e,x).$$

Now we use Theorem 2 for v' to obtain domination polynomial of $T_{n-2}+e$, then we have

$$D(T_{n-2}+e,x)=(1+x)D(T_{n-2},x)+xD(T_{n-3}+e,x)-(1+x)D(T_{n-3}+e,x).$$

Therefore the result follows.

2.2 Computation of $D(T_n,x)$ by counting the number of dominating sets

In this section we shall obtain a recurrence relation for the domination polynomial of T_n. For this purpose we count the number of dominating sets of T_n with cardinality k. In other words, we first find a two variables recursive formula for $d(T_n,k)$.

Recently by private communication, we found that the following result also appear in [6] but were proved independently.

Theorem 5 . *The number of dominating sets of T_n with cardinality k is given by*

$$d(T_n,k)=2d(T_{n-1},k-1)+d(T_{n-1},k-2)+d(T_{n-2},k-1)+d(T_{n-2},k-2).$$

Proof. We shall make a dominating set of T_n with cardinality k which we denote it by T_n^k. We consider all cases:

Case 1. If T_n^k contains both of v and w, then we have $\mathsf{T}_n^k=\mathsf{T}_{n-1}^{k-2}\cup\{v,w\}$. In this case we have $d(T_n,k)=d(T_{n-1},k-2)$.

Case 2. If T_n^k contains only v or w (say v), then we have $\mathsf{T}_n^k=\mathsf{T}_{n-1}^{k-1}\cup\{v\}$. In this case we have $d(T_n,k)=2d(T_{n-1},k-1)$.

Case 3. If T_n^k contains none of v and w, then we can construct T_n^k by T_{n-2}^{k-1} or T_{n-2}^{k-2} as shown in Fig. 3. In this case we have $d(T_n,k)=d(T_{n-2},k-1)+d(T_{n-2},k-2)$.

By adding all contributions we obtain the recurrence for $d(T_n,k)$.

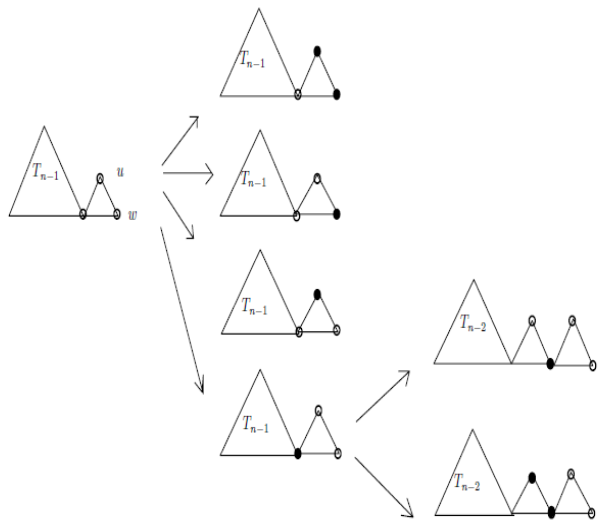

Fig. 3. Recurrence relation for $d(T_n,k)$.

Corollary 1 . *For every $n\ge 3$,*

$$D(T_n,x)=(x^2+2x)D(T_{n-1},x)+(x^2+x)D(T_{n-2},x).$$

Proof. It follows from Theorem 5 and the definition of the domination polynomial.

We mention here the Hosoya index of a graph G is the total number of matchings of G and the Merrifield-Simmons index is the total number of its independent sets. Motivation by these indices, we are interested to count the total number of dominating set of a graph which is equal to $D(G,1)$. Here we present a recurrence relation to the total number of the chain triangular cactus.

Theorem 6 . *The enumerating sequence $\{t_n\}$ for the number of dominating sets in T_n $(n\ge 2)$ is*

$$t_n=3t_{n-1}+2t_{n-2}$$

with initial values $t_0=2$, $t_1=7$.

Proof. Since $t_n=D(T_n,1)$, it follows from Corollary 1.

3. Counting the number of dominating sets of chains of squares

By replacing triangles in the definitions of triangular cactus, by cycles of length 4 we obtain cacti whose every block is C_4. We call such cacti, square cacti. An example of a square cactus chain is shown in Fig. 4. We see that the internal squares may differ in the way they connect to their neighbors. If their cut-vertices are adjacent, we say that such a square is an ortho-square; if the cut-vertices are not adjacent, we call the square a para-square.

3.1 Domination polynomial of para-chain square cactus graphs

In this subsection we consider a para-chain of length n, Q_n, as shown in Fig. 4. We shall obtain a recurrence relation for the domination polynomial of Q_n. As usual we denote the number of dominating sets of Q_n by $d(Q_n,k)$. The following theorem gives a recurrence relation for $D(Q_n,x)$.

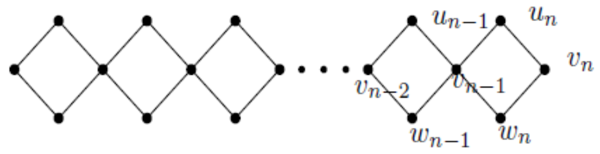

Fig. 4. Para-chain square cactus graphs.

We need the following Lemma for finding domination polynomial of the Q_n.

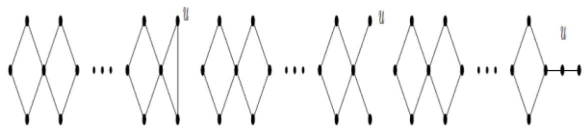

Fig. 5. Graphs Q_n^{Δ}, $Q_{n'}$ and $Q_n(2)$, respectively.

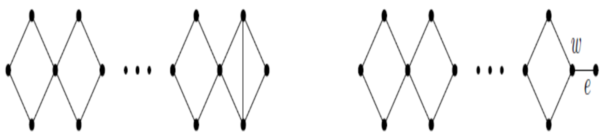

Fig. 6. Graphs $(Q_n+e)/w$ and Q_n+e, respectively.

Lemma 1 . *For graphs in figures 5 and 6 have:*

(i) $D(Q_n^{\Delta},x)=(1+x)D(Q_n+e,x)+xD(Q_{n-1}',x)$, where $D(Q_0^{\Delta},x)=x^3+3x^2+3x$.

(ii) $D(Q_n(2),x)=x(D(Q_n+e,x)+D(Q_n,x)+D(Q_{n-1}',x))$, where $D(Q_0(2),x)=x^3+3x^2+x$.

(iii) $D(Q_{n'},x)=(1+x)D(Q_n+e,x)-xD(Q_{n-1}',x)$, where $D(Q_{0'},x)=x^3+3x^2+x$.

(iv) $D(Q_n+e,x)=x(D(Q_n,x)+D(Q_{n-1},x))+xD(Q_{n-1}',x)+2x^2D(Q_{n-2}',x)$

where $D(Q_1+e,x)=x^5+5x^4+9x^3+4x^2$.

Proof. The proof of parts (i) and (ii) follow from Theorems 1 and 2 for vertex u in graphs Q_n^{Δ} and $Q_n(2)$, respectively. Note that in these cases $p_u(G,x)=0$.

(iii) We use Theorems 1 and 2 for vertex u to obtain domination polynomial of $Q_{n'}$, then we have

$$D(Q_{n'},x)=(1+x)D(Q_n+e,x)+x^2D(Q_{n-1}',x)-(1+x)xD(Q_{n-1}',x)$$
$$=(1+x)D(Q_n+e,x)-x^2D(Q_{n-1}',x).$$

(iv) We use Theorems 1 and 2 for vertex w to obtain domination polynomial of Q_n+e, as shown in Fig. 6 then we have

$$D(Q_n+e,x)=xD((Q_n+e)/w,x)+xD(Q_{n-1}',x)+xD(Q_{n-1},x).$$

Now consider the graph $(Q_n+e)/w$ as shown in Fig. 6. We use Theorems 1 and 3 for $e=\{u,v\}$ to obtain $D((Q_n+e)/w,x)$, then we have

$$D((Q_n+e)/w,x)=D(Q_n,x)+\frac{x}{x-1}[D(Q_{n-1}^{\Delta},x)+D(Q_{n-1}^{\Delta},x)-(Q_{n-1}^{\Delta},x)-D(Q_{n-1}^{\Delta},x)-D(Q_{n-2}',x)-D(Q_{n-2}',x)+xD(Q_{n-2}',x)+xD(Q_{n-2}',x)]$$
$$=D(Q_n,x)+2xD(Q_{n-2}',x).$$

Therefore the result follows.

Theorem 7. *The domination polynomial of para-chain Q_n is given by*

$$D(Q_n,x)=(x^3+2x^2+x)D(Q_{n-1},x)+(x^3+2x^2)D(Q_{n-2},x)+(x^3+3x^2)D(Q_{n-2}',x)+(2x^4+4x^3)D(Q_{n-3}',x),$$

with initial conditions $D(Q_1,x)=x^4+4x^3+6x^2$ and $D(Q_2,x)=x^7+7x^6+21x^5+29x^4+15x^3$.

Proof. Consider the labeled Q_n as shown in Figure 4. We use Theorems 1 and 2 for vertex u_n to obtain the domination polynomial of Q_n. We have

$$D(Q_n,x)=xD(Q_{n-1}^{\Delta},x)+D(Q_{n-1}(2),x)+x^2D(Q_{n-2}',x)-(1+x)xD(Q_{n-2}',x)$$
$$=xD(Q_{n-1}^{\Delta},x)+D(Q_{n-1}(2),x)-xD(Q_{n-2}',x). \quad (2)$$

Therefore by parts $(i),(ii)$ and (iv) of Lemma 1 and equation (2) we have

$$D(Q_n,x)=x((1+x)D(Q_{n-1}+e,x)+xD(Q_{n-2}',x))+x(D(Q_{n-1}+e,x)$$

$$+D(Q_{n-1},x)+D(Q_{n-2}',x))-xD(Q_{n-2}',x)$$
$$=(x^2+2x)D(Q_{n-1}+e,x)+x^2D(Q_{n-2}',x)+xD(Q_{n-1},x)$$
$$=(x^2+2x)[x(D(Q_{n-1},x)+D(Q_{n-2},x))+xD(Q_{n-2}',x)$$
$$+2x^2D(Q_{n-3}',x)]+x^2D(Q_{n-2}',x)+xD(Q_{n-1},x)$$
$$=(x^3+2x^2+x)D(Q_{n-1},x)+(x^3+2x^2)D(Q_{n-2},x)$$
$$+(x^3+3x^2)D(Q_{n-2}',x)+(2x^4+4x^3)D(Q_{n-3}',x).$$

3.2 Domination polynomial of ortho-chain square cactus graphs

In this subsection we consider a ortho-chain of length n, O_n, as shown in Fig. 7. We shall obtain a recurrence relation for the domination polynomial of O_n.

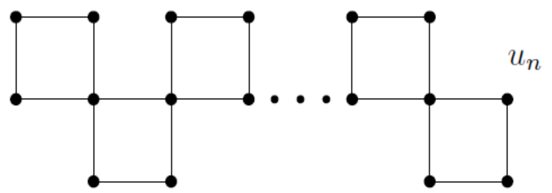

Fig. 7. Labeled ortho-chain square O_n.

We need the following Lemma for finding domination polynomial of the O_n.

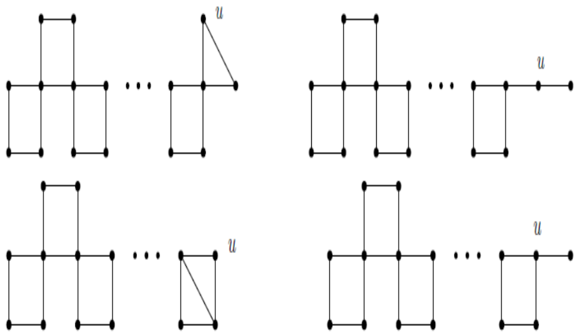

Fig. 8. Graphs O_n^Δ, $O_n(2)$, $O_{n'}$ and O_n+e, respectively.

Lemma 2 . *For graphs in figure 8 we have:*

(i) $D(O_n^\Delta,x)=(1+x)D(O_n+e,x)+xD(O_{n-1}(2),x)$, where $D(O_0^\Delta,x)=x^3+3x^2+3x$.

(ii) $D(O_n(2),x)=x(D(O_n+e,x)+D(O_n,x)+D(O_{n-1}(2),x))$ where $D(O_0(2),x)=x^3+3x^2+x$.

(iii) $D(O_{n'},x)=(1+x)D(O_n^\Delta,x)-xD(O_{n-1}(2),x)$, where $D(O_{0'},x)=x^4+4x^3+6x^2+2x$.

(iv) $D(O_n+e,x)=xD(O_{n'},x)+xD(O_{n-1}(2),x)+x^2D(O_{n-2}(2),x)$ where $D(O_1+e,x)=x^5+5x^4+9x^3+4x^2$.

Proof. The proof of parts $(i),(ii)$ and (iv) follow from Theorems 1 and 2 for vertex u in graphs $O_n^\Delta, O_n(2)$ and O_n+e, respectively. Note that in these cases $p_u(G,x)=0$.

(iii) We use Theorems 1 and 2 for u in graphs $O_{n'}$. Since $O_{n'}/u$ is isomorphic to $O_{n'}-u$ and $p_u(G,x)=xD(O_{n-1}(2),x)$. So we have the result.

Theorem 8 . *The domination polynomial of para-chain O_n is given by*

$$D(O_n,x)=xD(O_{n-1},x)+(x^2+2x)D(O_{n-1}+e,x)+x^2D(O_{n-2}(2),x),$$

with initial condition $D(O_1,x)=x^4+4x^3+6x^2$.

Proof. Consider the labeled O_n as shown in Figure 7. We use Theorems 1 and 2 for vertex u_n to obtain domination polynomial of O_n, then we have

$$D(O_n,x)=xD(O_{n-1}^\Delta,x)+D(O_{n-1}(2),x)+$$
$$x^2D(O_{n-2}(2),x)-(1+x)xD(O_{n-2}(2),x)$$
$$=xD(O_{n-1}^\Delta,x)+D(O_{n-1}(2),x)-xD(O_{n-2}(2),x).$$

Therefore by parts (i) and (ii) of Lemma 2 and this equation we have

$$D(O_n,x)=x((1+x)D(O_{n-1}+e,x)+xD(O_{n-2}(2),x))+$$
$$x(D(O_{n-1}+e,x)$$
$$+D(O_{n-1},x)+D(O_{n-2}(2),x))-xD(O_{n-2}(2),x)$$
$$=(x^2+2x)D(O_{n-1}+e,x)+x^2D(O_{n-2}(2),x)+$$
$$xD(O_{n-1},x).$$

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