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Cubic–quartic optical soliton perturbation with Gerdjikov–Ivanov equation by sine-Gordon equation approach

YAKUP YIDIRIM[1], ANJAN BISWAS[2,3,4,5,*], HASHIM M. ALSHEHRI[3], MILIVOJ R. BELIC[6]

[1]Department of Mathematics, Faculty of Arts and Sciences, Near East University, 99138 Nicosia, Cyprus

[2]Department of Applied Mathematics, National Research Nuclear University, 31 Kashirskoe Hwy, Moscow–115409, Russian Federation

[3]Mathematical Modeling and Applied Computation (MMAC) Research Group, Department of Mathematics, King Abdulaziz University, Jeddah–21589, Saudi Arabia

[4]Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Medunsa–0204, Pretoria, South Africa

[5]Department of Applied Sciences, Cross–Border Faculty, Dunarea de Jos University of Galati, 111 Domneasca Street, Galati–800201, Romania

[6]Institute of Physics Belgrade, Pregrevica 118, 11080 Zemun, Serbia

This paper recovers cubic–quartic optical solitons for the perturbed Gerdjikov–Ivanov equation. The scalar case and birefringent fibers are both covered. The perturbation terms appear with maximum permissible intensity. A complete spectrum of soliton solutions are recovered and listed.

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1. Introduction

The propagation dynamics of solitons through an optical fiber, PCF or metamaterials is governed by the well–known nonlinear Schrödinger's equation (NLSE). However, there exists three forms of derivative NLSE that is also studied in this context although predominantly these forms mainly appear in Plasma Physics. One of the three forms that would be studied in the current work is the Gerdjikov–Ivanov (GI) equation [1–5]. This model is being considered with cubic–quartic (CQ) dispersive effects as opposed to the usual chromatic–dispersion (CD) being the source of dispersion. If it so happens that CD runs low, then the cumulative effect of third-order dispersion (3OD) and fourth–order dispersion (4OD) compensates for this low count. Hence, with this newly developed concept, the GI equation with CQ dispersion will be integrated in presence of perturbation terms using the sine–Gordon equation method, for the first time, in this paper. These perturbation terms appear with maximum allowable intensity. Both, the scalar case and birefringent fibers are taken into account. A full spectrum of soliton solutions are recovered from this algorithm that are enumerated in this paper. These solutions also come with parameter constraints that are also presented.

2. Scalar model

The CQ-GI equation with nonlinear perturbation terms in polarization preserving reads as:

$$iu_t + iau_{xxx} + bu_{xxxx} + c|u|^4u + idu^2u_x^* = i[\alpha u_x + \lambda(|u|^{2m}u)_x + \mu(|u|^{2m})_x u], \tag{1}$$

where x and t are the non-dimensional distance and time in dimensionless form respectively. The first term is linear temporal evolution and $i = \sqrt{-1}$. The complex valued function $u(x,t)$ represents optical solitons in polarization-preserving fibers. The constants a and b are respectively the coefficients of 3OD and 4OD while the coefficients c and d constitutes quintic nonlinearity and nonlinear dispersion term respectively. Lastly, α, λ and μ are respectively the coefficients of inter-modal dispersion, self-steepening term and higher-order dispersion.

To obtain optical solitons with the perturbed CQ-GI equation in polarization-preserving fibers, we assume the traveling wave transformation as

$$u(x,t) = U(\xi)e^{i\varphi(x,t)},$$

$$\xi = x - vt, \tag{2}$$

$$\varphi(x,t) = -\kappa x + \omega t + \theta_0,$$

where κ, v, ω and θ_0 are respectively the frequency, velocity, wave number and phase constant of the soliton. Also, the function $U(\xi)$ is the amplitude of the soliton while the function $\varphi(x,t)$ is the phase component of the soliton.

Inserting the traveling wave transformation (2) into the perturbed CQ-GI equation (1) leads to the ordinary differential equation

$$bU^{(iv)} + 6b\kappa^2U'' - (\omega + 3b\kappa^4 + \alpha\kappa)U$$
$$-\kappa(d+\lambda)U^3 + cU^5 = 0, \quad (3)$$

along with the parameter constraints

$$m = 1, \quad (4)$$

$$a = 4b\kappa, \quad (5)$$

$$d - 3\lambda - 2\mu = 0, \quad (6)$$

$$v = -\alpha - 8b\kappa^3. \quad (7)$$

Eq. (3) can be integrated to determine the soliton profile while Eq. (7) gives the velocity of the soliton. According to the sine-Gordon equation approach, Eq. (3) holds the formal solution

$$U(\xi) = \sum_{i=1}^{N} \cos^{i-1}(V(\xi)) \begin{bmatrix} B_i\sin(V(\xi)) \\ A_i\cos(V(\xi)) \end{bmatrix} + A_0, \quad (8)$$

$$V'(\xi) = \sin\big(V(\xi)\big), \quad (9)$$

$$\begin{aligned} \sin\big(V(\xi)\big) &= \ sech\ (\xi), \\ \sin\big(V(\xi)\big) &= \mathrm{i}\ csh\ (\xi), \\ \cos\big(V(\xi)\big) &= \tanh(\xi), \\ \cos\big(V(\xi)\big) &= \coth(\xi), \end{aligned} \quad (10)$$

where A_i and B_i are constants and the value of the integer N is identified by applying the balance principle between the non-linear term and greatest order derivative term. Balancing $U^{(iv)}$ with U^5 in Eq. (3) gives $N = 1$. Thus, Eq. (8) reduces to

$$U(\xi) = A_0 + B_1\sin\big(V(\xi)\big)$$
$$+A_1\cos\big(V(\xi)\big). \quad (11)$$

Inserting Eq. (11) along with Eq. (9) into Eq. (3) gives rise to the following soliton solutions:

Case-1:

$$b = -\frac{3\kappa^2(d+\lambda)^2}{2c(3\kappa^2-10)^2}, \qquad A_0 = 0,$$

$$A_1 = \pm\sqrt{-\frac{6\kappa(d+\lambda)}{c(3\kappa^2-10)}}, \qquad B_1 = 0,$$

$$\omega = -\frac{\kappa}{2c(3\kappa^2-10)^2}(-9d^2\kappa^5 - 18d\kappa^5\lambda$$
$$-9\kappa^5\lambda^2 + 18\alpha c\kappa^4 - 36d^2\kappa^3 - 72d\kappa^3\lambda$$
$$-36\kappa^3\lambda^2 - 120\alpha c\kappa^2 + 48d^2\kappa$$
$$+96d\kappa\lambda + 48\kappa\lambda^2 + 200\alpha c). \quad (12)$$

Substituting Eq. (12) along with Eq. (10) into Eq. (11) causes to the dark soliton solution

$$u(x,t) = \pm\sqrt{-\frac{6\kappa(d+\lambda)}{c(3\kappa^2-10)}}$$
$$\times \tanh(x + (\alpha + 8b\kappa^3)t)e^{i(-\kappa x+\omega t+\theta_0)}, \quad (13)$$

and the singular soliton solution

$$u(x,t) = \pm\sqrt{-\frac{6\kappa(d+\lambda)}{c(3\kappa^2-10)}}$$
$$\times \coth(x + (\alpha + 8b\kappa^3)t)e^{i(-\kappa x+\omega t+\theta_0)}. \quad (14)$$

The dark soliton solution (13) and the singular soliton solution (14) are yielded by the constraint

$$\kappa c(d+\lambda)(3\kappa^2-10) < 0.$$

Case-2:

$$b = -\frac{3\kappa^2(d+\lambda)^2}{2c(3\ \kappa^2+5)^2}, \qquad A_0 = 0,$$

$$A_1 = 0, \qquad B_1 = \pm\sqrt{\frac{6\kappa(d+\lambda)}{c(3\kappa^2+5)}},$$

$$\omega = -\frac{\kappa}{2c(3\kappa^2+5)^2}(-9d^2\kappa^5 - 18d\kappa^5\lambda$$
$$-9\kappa^5\lambda^2 + 18\alpha c\kappa^4 + 18d^2\kappa^3 + 36d\kappa^3\lambda$$
$$+18\kappa^3\lambda^2 + 60\alpha c\kappa^2 + 3d^2\kappa$$
$$+6d\kappa\lambda + 3\kappa\lambda^2 + 50\alpha c). \quad (15)$$

Inserting Eq. (15) along with Eq. (10) into Eq. (11) leads to the bright soliton solution

$$u(x,t) = \pm\sqrt{\frac{6\kappa(d+\lambda)}{c(3\kappa^2+5)}}$$

$$\times\ sech\ (x+(\alpha+8b\kappa^3)t)e^{i(-\kappa x+\omega t+\theta_0)}, \tag{16}$$

and the singular soliton solution

$$u(x,t) = \pm\sqrt{-\frac{6\kappa(d+\lambda)}{c(3\kappa^2+5)}}$$

$$\times\ csch\ (x+(\alpha+8b\kappa^3)t)e^{i(-\kappa x+\omega t+\theta_0)}. \tag{17}$$

The bright soliton solution (16) is given by the constraint

$$\kappa c(d+\lambda) > 0,$$

while the singular soliton solution (17) is derived by the constraint

$$\kappa c(d+\lambda) < 0.$$

Case-3:

$$b = -\frac{6\kappa^2(d+\lambda)^2}{c(6\kappa^2-5)^2}, \qquad A_0 = 0,$$

$$A_1 = \pm\sqrt{-\frac{3\kappa(d+\lambda)}{c(6\kappa^2-5)}},$$

$$B_1 = \pm\sqrt{\frac{3\kappa(d+\lambda)}{c(6\kappa^2-5)}},$$

$$\omega = -\frac{\kappa}{c(6\kappa^2-5)^2}(-18d^2\kappa^5 - 36d\kappa^5\lambda$$

$$-18\kappa^5\lambda^2 + 36\alpha c\kappa^4 - 18d^2\kappa^3 - 36d\kappa^3\lambda$$

$$-18\kappa^3\lambda^2 - 60\alpha c\kappa^2 + 6d^2\kappa$$

$$+12d\kappa\lambda + 6\kappa\lambda^2 + 25\alpha c). \tag{18}$$

Substituting Eq. (18) along with Eq. (10) into Eq. (11) yields the combo singular soliton solution

$$u(x,t) = \pm\sqrt{-\frac{3\kappa(d+\lambda)}{c(6\kappa^2-5)}}$$

$$\times\begin{Bmatrix}\coth(x+(\alpha+8b\kappa^3)t)\\ +\ csch\ (x+(\alpha+8b\kappa^3)t)\end{Bmatrix}$$

$$\times\ e^{i(-\kappa x+\omega t+\theta_0)}. \tag{19}$$

The combo singular soliton solution (19) is retrieved by the constraint

$$\kappa c(d+\lambda)(6\kappa^2-5) < 0.$$

Case-4:

$$\kappa = \pm\frac{\sqrt{30}}{6}, \qquad b = \frac{(d+\lambda)^2}{54c\kappa^2}, \qquad A_0 = 0,$$

$$\omega = -\frac{648\alpha c\kappa^3 + 133d^2 + 266d\lambda + 133\lambda^2}{648c\kappa^2},$$

$$A_1 = \pm\sqrt{\frac{d+\lambda}{3c\kappa}},\ B_1 = \pm\sqrt{\frac{d+\lambda}{3c\kappa}}. \tag{20}$$

Inserting Eq. (20) along with Eq. (10) into Eq. (11) gives the combo dark-bright soliton solution

$$u(x,t) = \pm\sqrt{\frac{d+\lambda}{3c\kappa}}$$

$$\times\begin{Bmatrix}\tanh(x+(\alpha+8b\kappa^3)t)\\ +\ sech\ (x+(\alpha+8b\kappa^3)t)\end{Bmatrix}$$

$$\times\ e^{i(-\kappa x+\omega t+\theta_0)}. \tag{21}$$

The combo dark-bright soliton solution (21) is yielded by the constraint

$$\kappa c(d+\lambda) > 0.$$

3. Birefringent fibers

The CQ-GI equation with nonlinear perturbation terms in birefringent fibers reads as:

$$iq_t + ia_1q_{xxx} + b_1q_{xxxx}$$

$$+(c_1|q|^4 + d_1|q|^2|r|^2 + e_1|r|^4)q$$

$$+i(f_1q^2 + g_1r^2 + h_1qr)q_x^*$$

$$= i[\alpha_1q_x + \lambda_1(|q|^2q)_x + \mu_1(|q|^2)_xq], \tag{22}$$

$$ir_t + ia_2r_{xxx} + b_2r_{xxxx}$$

$$+(c_2|r|^4 + d_2|r|^2|q|^2 + e_2|q|^4)r$$

$$+i(f_2r^2 + g_2q^2 + h_2rq)r_x^*$$

$$= i[\alpha_2r_x + \lambda_2(|r|^2r)_x + \mu_2(|r|^2)_xr], \tag{23}$$

where the complex valued functions $q(x,t)$ and $r(x,t)$ account for optical solitons in birefringent fibers. For $l = 1,2$, a_l and b_l are respectively the coefficients of 3OD and 4OD and c_l represent the coefficients of self-phase modulation (SPM) while d_l and e_l account for the coefficients of cross-phase modulation (XPM). Also, f_l, g_l and h_l represent the coefficients of nonlinear dispersion. Lastly, α_l, λ_l and μ_l are respectively the coefficients of inter-modal dispersion, self-steepening term and higher-order dispersion.

To obtain optical solitons with the perturbed CQ-GI equation in birefringent fibers, we assume the traveling wave transformations as

$$q(x,t) = U_1(\xi)e^{i\varphi(x,t)},$$
$$r(x,t) = U_2(\xi)e^{i\varphi(x,t)},$$
$$\xi = x - vt, \tag{24}$$
$$\varphi(x,t) = -\kappa x + \omega t + \theta_0.$$

Inserting the traveling wave transformations (24) into the governing system (22) and (23) yields the real part

$$b_l U_l^{(iv)} + (3\kappa a_l - 6\kappa^2 b_l)U_l''$$
$$+(\kappa^4 b_l - \kappa\alpha_l - \kappa^3 a_l - \omega)U_l$$
$$-\kappa(f_l + \lambda_l)U_l^3 - \kappa h_l U_l^2 U_{\tilde{l}} - \kappa g_l U_l U_{\tilde{l}}^2$$
$$+c_l U_l^5 + e_l U_l U_{\tilde{l}}^4 + d_l U_l^3 U_{\tilde{l}}^2 = 0, \tag{25}$$

and the imaginary part

$$(4\kappa^3 b_l - v - \alpha_l - 3\kappa^2 a_l)U_l'$$
$$(a_l - 4\kappa b_l)U_l''' + h_l U_l U_{\tilde{l}} U_l' + g_l U_{\tilde{l}}^2 U_l'$$
$$+(f_l - 3\lambda_l - 2\mu_l)U_l^2 U_l' = 0, \tag{26}$$

where $l = 1,2$ and $\tilde{l} = 3 - l$. Eqs. (25) and (26) become

$$b_l U_l^{(iv)} + 6\kappa^2 b_l U_l''$$
$$-(\kappa\alpha_l + 3\kappa^4 b_l + \omega)U_l$$
$$-\kappa(f_l + \lambda_l + h_l + g_l)U_l^3$$
$$+(c_l + e_l + d_l)U_l^5 = 0, \tag{27}$$

by the constraints

$$U_{\tilde{l}} = U_l, \tag{28}$$

$$a_l = 4\kappa b_l, \tag{29}$$

$$v = -\alpha_l - 8\kappa^3 b_l, \tag{30}$$

$$h_l + g_l + f_l - 3\lambda_l - 2\mu_l = 0. \tag{31}$$

Balancing $U_l^{(iv)}$ with U_l^5 in Eq. (27) gives $N = 1$. Thus, Eq. (8) changes to

$$U_l(\xi) = A_0 + B_1\sin\big(V_l(\xi)\big)$$
$$+A_1\cos\big(V_l(\xi)\big). \tag{32}$$

Substituting Eq. (32) along with Eq. (9) into Eq. (27) causes to the following soliton solutions:

Case-1:

$$A_0 = 0, \qquad B_1 = 0,$$

$$A_1 = \pm\sqrt{-\frac{6\kappa(f_l + g_l + h_l + \lambda_l)}{(3\kappa^2 - 10)(c_l + d_l + e_l)}},$$

$$b_l = -\frac{3\kappa^2(f_l + g_l + h_l + \lambda_l)^2}{2(3\kappa^2 - 10)^2(c_l + d_l + e_l)},$$

$$\omega = \frac{\kappa}{2(3\kappa^2 - 10)^2(c_l + d_l + e_l)}(9\kappa^5 {f_l}^2$$
$$+18\kappa^5 f_l g_l + 18\kappa^5 f_l h_l + 18\kappa^5 f_l \lambda_l$$
$$+9\kappa^5 {g_l}^2 + 18\kappa^5 g_l h_l + 18\kappa^5 g_l \lambda_l$$
$$+9\kappa^5 {h_l}^2 + 18\kappa^5 h_l \lambda_l + 9\kappa^5 {\lambda_l}^2$$
$$-18\kappa^4 \alpha_l c_l - 18\kappa^4 \alpha_l d_l - 18\kappa^4 \alpha_l e_l$$
$$+36\kappa^3 {f_l}^2 + 72\kappa^3 f_l g_l + 72\kappa^3 f_l h_l$$
$$+72\kappa^3 f_l \lambda_l + 36\kappa^3 {g_l}^2 + 72\kappa^3 g_l h_l$$
$$+72\kappa^3 g_l \lambda_l + 36\kappa^3 {h_l}^2 + 72\kappa^3 h_l \lambda_l$$
$$+36\kappa^3 {\lambda_l}^2 + 120\kappa^2 \alpha_l c_l + 120\kappa^2 \alpha_l d_l$$
$$+120\kappa^2 \alpha_l e_l - 48\kappa {f_l}^2 - 96\kappa f_l g_l$$
$$-96\kappa f_l h_l - 96\kappa f_l \lambda_l - 48\kappa {g_l}^2$$
$$-96\kappa g_l h_l - 96\kappa g_l \lambda_l - 48\kappa {h_l}^2$$
$$-96\kappa h_l \lambda_l - 48\kappa {\lambda_l}^2$$
$$-200\alpha_l c_l - 200\alpha_l d_l - 200\alpha_l e_l). \tag{33}$$

Inserting Eq. (33) along with Eq. (10) into Eq. (32) yields the dark soliton solutions

$$q(x,t) = \pm\sqrt{-\frac{6\kappa(f_1 + g_1 + h_1 + \lambda_1)}{(3\kappa^2 - 10)(c_1 + d_1 + e_1)}}$$
$$\times \tanh(x + (\alpha_1 + 8\kappa^3 b_1)t)e^{i(-\kappa x + \omega t + \theta_0)}, \tag{34}$$

$$r(x,t) = \pm\sqrt{-\frac{6\kappa(f_2+g_2+h_2+\lambda_2)}{(3\kappa^2-10)(c_2+d_2+e_2)}}$$

$$\times \tanh(x+(\alpha_2+8\kappa^3 b_2)t)e^{i(-\kappa x+\omega t+\theta_0)}, \quad (35)$$

and the singular soliton solutions

$$q(x,t) = \pm\sqrt{-\frac{6\kappa(f_1+g_1+h_1+\lambda_1)}{(3\kappa^2-10)(c_1+d_1+e_1)}}$$

$$\times \coth(x+(\alpha_1+8\kappa^3 b_1)t)e^{i(-\kappa x+\omega t+\theta_0)}, \quad (36)$$

$$r(x,t) = \pm\sqrt{-\frac{6\kappa(f_2+g_2+h_2+\lambda_2)}{(3\kappa^2-10)(c_2+d_2+e_2)}}$$

$$\times \coth(x+(\alpha_2+8\kappa^3 b_2)t)e^{i(-\kappa x+\omega t+\theta_0)}. \quad (37)$$

The dark soliton solutions (34) and (35) and the singular soliton solutions (36) and (37) are retrieved by the constraint

$$\kappa(3\kappa^2-10)(c_l+d_l+e_l)(f_l+g_l+h_l+\lambda_l) < 0.$$

Case-2:

$$A_0 = 0, \qquad A_1 = 0,$$

$$B_1 = \pm\sqrt{\frac{6\kappa(f_l+g_l+h_l+\lambda_l)}{(3\kappa^2+5)(c_l+d_l+e_l)}},$$

$$b_l = -\frac{3\kappa^2(f_l+g_l+h_l+\lambda_l)^2}{2(3\kappa^2+5)^2(c_l+d_l+e_l)},$$

$$\omega = \frac{\kappa}{2(3\kappa^2+5)^2(c_l+d_l+e_l)}(9\kappa^5 f_l^2$$
$$+18\kappa^5 f_l g_l + 18\kappa^5 f_l h_l + 18\kappa^5 f_l\lambda_l$$
$$+9\kappa^5 g_l^2 + 18\kappa^5 g_l h_l + 18\kappa^5 g_l\lambda_l$$
$$+9\kappa^5 h_l^2 + 18\kappa^5 h_l\lambda_l + 9\kappa^5\lambda_l^2$$
$$-18\kappa^4\alpha_l c_l - 18\kappa^4\alpha_l d_l - 18\kappa^4\alpha_l e_l$$
$$-18\kappa^3 f_l^2 - 36\kappa^3 f_l g_l - 36\kappa^3 f_l h_l$$
$$-36\kappa^3 f_l\lambda_l - 18\kappa^3 g_l^2 - 36\kappa^3 g_l h_l$$
$$-36\kappa^3 g_l\lambda_l - 18\kappa^3 h_l^2 - 36\kappa^3 h_l\lambda_l$$
$$-18\kappa^3\lambda_l^2 - 60\kappa^2\alpha_l c_l - 60\kappa^2\alpha_l d_l$$
$$-60\kappa^2\alpha_l e_l - 3\kappa f_l^2 - 6\kappa f_l g_l - 6\kappa f_l h_l$$
$$-6\kappa f_l\lambda_l - 3\kappa g_l^2 - 6\kappa g_l h_l - 6\kappa g_l\lambda_l$$
$$-3\kappa h_l^2 - 6\kappa h_l\lambda_l - 3\kappa\lambda_l^2$$
$$-50\alpha_l c_l - 50\alpha_l d_l - 50\alpha_l e_l). \quad (38)$$

Substituting Eq. (38) along with Eq. (10) into Eq. (32) gives the bright soliton solutions

$$q(x,t) = \pm\sqrt{\frac{6\kappa(f_1+g_1+h_1+\lambda_1)}{(3\kappa^2+5)(c_1+d_1+e_1)}}$$

$$\times \mathrm{sech}(x+(\alpha_1+8\kappa^3 b_1)t)e^{i(-\kappa x+\omega t+\theta_0)}, \quad (39)$$

$$r(x,t) = \pm\sqrt{\frac{6\kappa(f_2+g_2+h_2+\lambda_2)}{(3\kappa^2+5)(c_2+d_2+e_2)}}$$

$$\times\ sech\ (x+(\alpha_2+8\kappa^3 b_2)t)e^{i(-\kappa x+\omega t+\theta_0)}, \quad (40)$$

and the singular soliton solutions

$$q(x,t) = \pm\sqrt{-\frac{6\kappa(f_1+g_1+h_1+\lambda_1)}{(3\kappa^2+5)(c_1+d_1+e_1)}}$$

$$\times\ csch\ (x+(\alpha_1+8\kappa^3 b_1)t)e^{i(-\kappa x+\omega t+\theta_0)}, \quad (41)$$

$$r(x,t) = \pm\sqrt{-\frac{6\kappa(f_2+g_2+h_2+\lambda_2)}{(3\kappa^2+5)(c_2+d_2+e_2)}}$$

$$\times\ csch\ (x+(\alpha_2+8\kappa^3 b_2)t)e^{i(-\kappa x+\omega t+\theta_0)}. \quad (42)$$

The bright soliton solutions (39) and (40) are given by the constraint

$$\kappa(c_l+d_l+e_l)(f_l+g_l+h_l+\lambda_l) > 0,$$

while the singular soliton solutions (41) and (42) are yielded by the constraint

$$\kappa(c_l+d_l+e_l)(f_l+g_l+h_l+\lambda_l) < 0.$$

Case-3:

$$A_0 = 0,$$

$$A_1 = \pm\sqrt{-\frac{3\kappa(f_l+g_l+h_l+\lambda_l)}{(6\kappa^2-5)(c_l+d_l+e_l)}},$$

$$B_1 = \pm\sqrt{\frac{3\kappa(f_l+g_l+h_l+\lambda_l)}{(6\kappa^2-5)(c_l+d_l+e_l)}},$$

$$b_l = -\frac{6\kappa^2(f_l+g_l+h_l+\lambda_l)^2}{(6\kappa^2-5)^2(c_l+d_l+e_l)},$$

$$\omega = \frac{\kappa}{(6\kappa^2-5)^2(c_l+d_l+e_l)}(18\kappa^5 {f_l}^2$$
$$+36\kappa^5 f_l g_l + 36\kappa^5 f_l h_l + 36\kappa^5 f_l \lambda_l$$
$$+18\kappa^5 {g_l}^2 + 36\kappa^5 g_l h_l + 36\kappa^5 g_l \lambda_l$$
$$+18\kappa^5 {h_l}^2 + 36\kappa^5 h_l \lambda_l + 18\kappa^5 {\lambda_l}^2$$
$$-36\kappa^4 \alpha_l c_l - 36\kappa^4 \alpha_l d_l - 36\kappa^4 \alpha_l e_l$$
$$+18\kappa^3 {f_l}^2 + 36\kappa^3 f_l g_l + 36\kappa^3 f_l h_l$$
$$+36\kappa^3 f_l \lambda_l + 18\kappa^3 {g_l}^2 + 36\kappa^3 g_l h_l$$
$$+36\kappa^3 g_l \lambda_l + 18\kappa^3 {h_l}^2 + 36\kappa^3 h_l \lambda_l$$
$$+18\kappa^3 {\lambda_l}^2 + 60\kappa^2 \alpha_l c_l + 60\kappa^2 \alpha_l d_l$$
$$+60\kappa^2 \alpha_l e_l - 6\kappa {f_l}^2 - 12\kappa f_l g_l$$
$$-12\kappa f_l h_l - 12\kappa f_l \lambda_l - 6\kappa {g_l}^2 - 6\kappa {h_l}^2$$
$$-12\kappa g_l h_l - 12\kappa g_l \lambda_l - 12\kappa h_l \lambda_l$$
$$-6\kappa {\lambda_l}^2 - 25\alpha_l c_l - 25\alpha_l d_l - 25\alpha_l e_l). \quad (43)$$

Inserting Eq. (43) along with Eq. (10) into Eq. (32) leads to the combo singular soliton solutions

$$q(x,t) = \pm\sqrt{-\frac{3\kappa(f_1+g_1+h_1+\lambda_1)}{(6\kappa^2-5)(c_1+d_1+e_1)}}$$
$$\times\begin{Bmatrix}\coth(x+(\alpha_1+8\kappa^3 b_1)t) \\ +\,csch\,(x+(\alpha_1+8\kappa^3 b_1)t)\end{Bmatrix}$$
$$\times e^{i(-\kappa x+\omega t+\theta_0)}, \quad (44)$$

$$r(x,t) = \pm\sqrt{-\frac{3\kappa(f_2+g_2+h_2+\lambda_2)}{(6\kappa^2-5)(c_2+d_2+e_2)}}$$
$$\times\begin{Bmatrix}\coth(x+(\alpha_2+8\kappa^3 b_2)t) \\ +\,csch\,(x+(\alpha_2+8\kappa^3 b_2)t)\end{Bmatrix}$$
$$\times e^{i(-\kappa x+\omega t+\theta_0)}. \quad (45)$$

The combo singular soliton solutions (44) and (45) are recovered by the constraint

$$\kappa(6\kappa^2-5)(c_l+d_l+e_l)(f_l+g_l+h_l+\lambda_l) < 0.$$

Case-4:

$$\kappa = \pm\frac{\sqrt{30}}{6}, \qquad A_0 = 0,$$

$$A_1 = \pm\sqrt{\frac{f_l+g_l+h_l+\lambda_l}{3\kappa(c_l+d_l+e_l)}},$$

$$B_1 = \pm\sqrt{\frac{f_l+g_l+h_l+\lambda_l}{3\kappa(c_l+d_l+e_l)}},$$

$$b_l = \frac{(f_l+g_l+h_l+\lambda_l)^2}{54(c_l+d_l+e_l)\kappa^2},$$

$$\omega = -\frac{1}{648(c_l+d_l+e_l)\kappa^2}(648\kappa^3\alpha_l c_l$$
$$+648\kappa^3\alpha_l d_l + 648\kappa^3 \alpha_l e_l + 133{f_l}^2$$
$$+266 f_l g_l + 266 f_l h_l + 266 f_l \lambda_l$$
$$+133{g_l}^2 + 266 g_l h_l + 266 g_l \lambda_l$$
$$+133{h_l}^2 + 266 h_l \lambda_l + 133{\lambda_l}^2). \quad (46)$$

Substituting Eq. (46) along with Eq. (10) into Eq. (32) yields the combo dark-bright soliton solutions

$$q(x,t) = \pm\sqrt{\frac{f_1+g_1+h_1+\lambda_1}{3\kappa(c_1+d_1+e_1)}}$$
$$\times\begin{Bmatrix}\tanh(x+(\alpha_1+8\kappa^3 b_1)t) \\ +\,sech\,(x+(\alpha_1+8\kappa^3 b_1)t)\end{Bmatrix}$$
$$\times e^{i(-\kappa x+\omega t+\theta_0)}, \quad (47)$$

$$r(x,t) = \pm\sqrt{\frac{f_2+g_2+h_2+\lambda_2}{3\kappa(c_2+d_2+e_2)}}$$
$$\times\begin{Bmatrix}\tanh(x+(\alpha_2+8\kappa^3 b_2)t) \\ +\,sech\,(x+(\alpha_2+8\kappa^3 b_2)t)\end{Bmatrix}$$
$$\times e^{i(-\kappa x+\omega t+\theta_0)}. \quad (48)$$

The combo dark-bright soliton solutions (47) and (48) are given by the constraint

$$\kappa(c_l+d_l+e_l)(f_l+g_l+h_l+\lambda_l) > 0. \quad (49)$$

4. Conclusions

This paper applied the concept of CQ solitons to the familiar GI equation and retrieved soliton solutions to this model in presence of perturbation terms that were considered with maximum allowable intensity. The scalar model together with birefringent fibers yielded a complete spectrum of soliton solutions that are exhibited with their respective existence criteria. The results truly open up an abundance of avenues for further investigations in this regard. An immediate thought is to nail the conservation laws followed by the corresponding soliton perturbation theory. Additional integration technologies would also give further and more fruitful insight into the model. Such research activities are all under way. This is just a foot in the door!

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*Corresponding author: biswas.anjan@gmail.com