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Dispersive optical solitons with Schrödinger-Hirota equation by traveling wave hypothesis

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This paper studies dispersive optical solitons, governed by Schrödinger-Hirota equation by the aid of traveling wave hypothesis. The spatio-temporal dispersion term is included, in addition to group velocity dispersion, to make the problem well-posed. Bright soliton solutions are retrieved along with constraint conditions for these solitons to exist. Both Kerr and power laws of nonlinearity are studied.

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1. Introduction

Optical solitons is an important area of research in the field of nonlinear optics [1-20]. Today, fiber optic communication links, across trans-oceanic and trans-continental distances, are achieved by means of optical solitons. Therefore, it is imperative to take a deeper look into the aspects of solitons, especially from a mathematical perspective. This enables to paint a better conceptual picture behind the physics of soliton propagation. Solitons are the outcome of a delicate balance between dispersion and nonlinearity.

This paper studies dispersive optical solitons where third order dispersion (3OD) is included in addition to usual group-velocity dispersion (GVD). This transforms the usual nonlinear Schrödinger's equation (NLSE) to Schrödinger-Hirota equation (SHE), after implementing Lie transform. Traveling wave hypothesis will be applied to obtain bright optical soliton solutions to SHE that will be considered with Kerr law and power law nonlinearities. There are several constraint conditions that will naturally emerge from the soliton solution structure. Therefore the bright solitons are guaranteed to exist when these constraints are in place.

2. Governing equation

The nonlinear evolution equation that models the propagation of solitons through optical fibers, with 3OD is the NLSE

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = -i\lambda u_{xxx} \tag{1}$$

Here λ is the coefficient of 3OD. The first term on the left hand side is the linear temporal evolution, while the second term represents GVD. Also, on left side, the third term accounts for Kerr law nonlinearity. The inclusion of 3OD is justified when GVD is low. Next, to study this equation in details, the following Lie symmetry is introduced:

$$q = u - 3i\lambda\left[u_x + 2u\int_{-\infty}^{x}|u(\xi)|^2 d\xi\right] \tag{2}$$

which transforms (1) to

$$iq_t + \frac{1}{2}q_{xx} + |q|^2 q + i\lambda(q_{xxx} + 6|q|^2 q_x) = 0 \tag{3}$$

after neglecting higher order terms [9, 12, 19]. Equation (3) is SHE with Kerr law nonlinearity. Therefore, SHE models transmission of dispersive optical solitons through nonlinear fibers. With arbitrary coefficients, SHE can be rewritten as

$$iq_t + aq_{xx} + c|q|^2 q + i(\gamma q_{xxx} + \sigma|q|^2 q_x) = 0 \tag{4}$$

Physically σ represents nonlinear dispersion. It was indicated during 2012, that GVD alone makes the governing model ill-posed [8, 15]. Therefore, it was proposed that inclusion of an additional dispersion term, namely the spatio-temporal dispersion (STD) introduces well-posedness [8, 15]. Hence, SHE with STD is

$$\begin{aligned} &iq_t + aq_{xx} + bq_{xt} + c|q|^2 q \\ &+ i(\gamma q_{xxx} + \sigma|q|^2 q_x) = 0 \end{aligned} \tag{5}$$

where the coefficient of b represents STD. Finally, in presence of perturbation terms, SHE with STD extends to

$$
\begin{aligned}
& iq_t + aq_{xx} + bq_{xt} + c|q|^2 q + i(\gamma q_{xxx} + \sigma |q|^2 q_x) \\
& = i\alpha q_x + i\lambda(|q|^2 q)_x + i\nu(|q|^2)_x q
\end{aligned} \tag{6}
$$

This paper will carry out the integration of the perturbed SHE with STD by implementing traveling wave hypothesis. This analysis will be detailed in next section.

3. Traveling wave hypothesis

The starting hypothesis for solving (6) by the aid of traveling waves is given by [1, 4]

$$q(x,t) = g(s)e^{i\phi(x,t)} = g(x - vt)e^{i\phi(x,t)} \tag{7}$$

where in (7), $g(x,t)$ represents the wave profile and $\phi(x,t)$ is the phase component of the soliton that is defined as

$$\phi(x,t) = -\kappa x + \omega t + \theta \tag{8}$$

and also

$$s = x - vt \tag{9}$$

Here, in (9) v represents the soliton velocity, while from phase component that is given by (8), κ is the constant soliton frequency, ω is soliton wave number that is also taken to be constant and θ is the center of phase or in other words phase constant for the soliton. The study will now be subdivided into two subsections that concentrates on Kerr law and power law media.

3.1 Kerr law

Substituting (7) into (6), and decomposing into real and imaginary parts lead to [1, 4]

$$
\begin{aligned}
& (a - \omega b v + 3\kappa\gamma)g'' \\
& = (\omega + b\omega\kappa + a\kappa^2 + \gamma\kappa^3 + \alpha\kappa)g \\
& - (\lambda\kappa - \sigma\kappa - c)g^3
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
& \gamma g''' = \begin{Bmatrix} v + \alpha + 2a\kappa \\ + b\omega(\omega + \kappa v) + 3\gamma\kappa^2 \end{Bmatrix} g' \\
& - (\sigma - 3\lambda - 2\nu) g^2 g'
\end{aligned} \tag{11}
$$

respectively, after simplification. The notations $g' = dg/ds$, $g'' = d^2 g/ds^2$ and so on are introduced.

To start with the real part equation, multiplying (10) by g' and integrating, yields, after choosing integration constant to be zero

$$(g')^2 = \frac{c + \sigma\kappa - \lambda\kappa}{2(a - bv\omega + 3\gamma\kappa)} g^2 (M^2 - g^2) \tag{12}$$

where

$$M^2 = \frac{2(\omega + \alpha\kappa - b\omega\kappa + a\kappa^2 + \gamma\kappa^3)}{c + \sigma\kappa - \lambda\kappa} \tag{13}$$

Separating variables and integrating (12) gives

$$
\begin{aligned}
& \sqrt{\frac{c + \sigma\kappa - \lambda\kappa}{2(a - bv\omega + 3\gamma\kappa)}}(x - vt) \\
& = \int \frac{dg}{g\sqrt{M^2 - g^2}}
\end{aligned} \tag{14}
$$

which implies

$$g(x,t) = A_1 \operatorname{sech}[B_1(x - vt)] \tag{15}$$

where the amplitude A_1 and the inverse width B_1 of the soliton are respectively

$$A_1 = \sqrt{\frac{2(\omega + \alpha\kappa - b\omega\kappa + a\kappa^2 + \gamma\kappa^3)}{c + \sigma\kappa - \lambda\kappa}} \tag{16}$$

and

$$B_1 = \sqrt{\frac{2(\omega + \alpha\kappa - b\omega\kappa + a\kappa^2 + \gamma\kappa^3)}{a - bv\omega + 3\gamma\kappa}} \tag{17}$$

Next, integrating the imaginary part equation (11) once, and again choosing the integration constant to be zero, since the search is for a soliton solution, leads to

$$
\begin{aligned}
& \gamma g'' = \{v + \alpha + 2a\kappa + b\omega(\omega + \kappa v) + 3\gamma\kappa^2\} g \\
& - (\sigma - 3\lambda - 2\nu)\frac{g^3}{3}
\end{aligned} \tag{18}
$$

Proceeding, after separation of variables, one recovers

$$\sqrt{\frac{\sigma-3\lambda-2\nu}{6\gamma}}(x-vt)=\int\frac{dg}{g\sqrt{N^2-g^2}} \tag{19}$$

where

$$N^2=\frac{6\left\{v+\alpha+2a\kappa+3\gamma\kappa^2-\omega b(\omega+\kappa v)\right\}}{\sigma-3\lambda-2\nu} \tag{20}$$

This gives

$$g(x,t)=A_2\operatorname{sech}[B_2(x-vt)] \tag{21}$$

where the amplitude A_2 and the inverse width B_2 of the soliton are respectively given by

$$A_2=\sqrt{\frac{6\begin{Bmatrix}v+\alpha+2a\kappa\\+3\gamma\kappa^2-\omega b(\omega+\kappa v)\end{Bmatrix}}{\sigma-3\lambda-2\nu}} \tag{22}$$

$$B_2=\sqrt{\frac{2\begin{Bmatrix}v+\alpha+2a\kappa\\+3\gamma\kappa^2-\omega b(\omega+\kappa v)\end{Bmatrix}}{\gamma}} \tag{23}$$

respectively. Since (15) and (21) represent the same soliton, equating the two amplitudes A_1 and A_2 gives

$$\begin{aligned}&\omega+\alpha\kappa-b\omega\kappa+a\kappa^2+\gamma\kappa^3\\&=\frac{3\begin{Bmatrix}v+\alpha+2a\kappa+3\gamma\kappa^2\\-b\omega(\omega+\kappa v)\end{Bmatrix}(c+\sigma\kappa-\lambda\kappa)}{\sigma-3\lambda-2\nu}\end{aligned} \tag{24}$$

and then equating the two widths B_1 and B_2 leads to

$$\begin{aligned}&\omega+\alpha\kappa-b\omega\kappa+a\kappa^2+\gamma\kappa^3\\&=\frac{\begin{pmatrix}a-bv\omega\\+3\gamma\kappa\end{pmatrix}\begin{Bmatrix}v+\alpha+2a\kappa+3\gamma\kappa^2\\-\omega b(\omega+\kappa v)\end{Bmatrix}}{\gamma}\end{aligned} \tag{25}$$

Finally, from (24) and (25), equating the two right hand sides, reveals

$$\begin{aligned}&3\gamma(c+\sigma\kappa-\lambda\kappa)\\&=(\sigma-3\lambda-2\nu)(a-bv\omega+3\gamma\kappa)\end{aligned} \tag{26}$$

which serves as a constraint relation between the coefficients and soliton parameters of the SHE with Kerr law nonlinearity. Thus, finally, the 1-soliton solution of (6) is

$$q(x,t)=A\operatorname{sech}[B(x-vt)]e^{i(-\kappa x+\omega t+\theta)} \tag{27}$$

where the amplitude and the inverse width of the soliton is given by the pair (16)-(17) or (22)-(23). The wave number of the soliton is from (24) or (25). This leads to the constraint condition given by (26). Additional constraint conditions are

$$\begin{aligned}&(c+\sigma\kappa-\lambda\kappa)\\&\times(\omega+\alpha\kappa-b\omega\kappa+a\kappa^2+\gamma\kappa^3)>0\end{aligned} \tag{28}$$

$$\begin{aligned}&(a-bv\omega+3\gamma\kappa)\\&\times(\omega+\alpha\kappa-b\omega\kappa+a\kappa^2+\gamma\kappa^3)>0\end{aligned} \tag{29}$$

$$\begin{aligned}&(\sigma-3\lambda-2\nu)\\&\times\left\{v+\alpha+2a\kappa+3\gamma\kappa^2-\omega b(\omega+\kappa v)\right\}>0\end{aligned} \tag{30}$$

and

$$\gamma\left\{v+\alpha+2a\kappa+3\gamma\kappa^2-\omega b(\omega+\kappa v)\right\}>0 \tag{31}$$

that follow from relations (16)-(17) and (22)-(23), respectively. These conditions guarantee the existence of bright solitons for SHE with STD in Kerr law medium.

3.2 Power law

For power law nonlinearity, perturbed SHE with STD extends to

$$\begin{aligned}&iq_t+aq_{xx}+bq_{xt}+c|q|^{2n}q+i(\gamma q_{xxx}+\sigma|q|^{2n}q_x)\\&=i\alpha q_x+i\lambda(|q|^{2n}q)_x+i\nu(|q|^{2n})_x q\end{aligned} \tag{32}$$

where the power-law nonlinearity factor n kicks in. It was proved earlier that the restriction $0<n<2$ must remain valid to avoid self-focusing singularity [14]. It is clear that Kerr law nonlinearity falls out upon setting $n=1$ in (32).

In order to integrate (32), the same hypothesis given by (7)-(9) is substituted into (32) and subsequently decomposed into real and imaginary parts. These respectively lead to

$$(a-\omega bv+3\kappa\gamma)g'' = (\omega+b\omega\kappa+a\kappa^2+\gamma\kappa^3+\alpha\kappa)g - (\lambda\kappa-\sigma\kappa-c)g^{2n+1} \quad (33)$$

and

$$\gamma g''' = \left\{\begin{array}{l} v+\alpha+2a\kappa \\ +b\omega(\omega+\kappa v)+3\gamma\kappa^2 \end{array}\right\} g' - (\sigma-3\lambda-2v)g^{2n}g' \quad (34)$$

after simplification.

To start with the real part equation, multiplying (33) by g' and integrating, yields, after simplification and once again choosing integration constant to be zero

$$(g')^2 = \frac{c+\sigma\kappa-\lambda\kappa}{(n+1)(a-bv\omega+3\gamma\kappa)} g^2(M^2-g^{2n}) \quad (35)$$

where

$$M^2 = \frac{(n+1)(\omega+\alpha\kappa-b\omega\kappa+a\kappa^2+\gamma\kappa^3)}{c+\sigma\kappa-\lambda\kappa} \quad (36)$$

Separating variables and integrating (35) gives

$$\sqrt{\frac{c+\sigma\kappa-\lambda\kappa}{(n+1)(a-bv\omega+3\gamma\kappa)}}(x-vt) = \int \frac{dg}{g\sqrt{M^2-g^{2n}}} \quad (37)$$

which leads to

$$g(x,t) = A_1 \operatorname{sech}^{\frac{1}{n}}[B_1(x-vt)] \quad (38)$$

where the amplitude A_1 and the inverse width B_1 of the soliton are respectively given by

$$A_1 = \left[\frac{(n+1)\left(\begin{array}{l} \omega+\alpha\kappa-b\omega\kappa \\ +a\kappa^2+\gamma\kappa^3 \end{array}\right)}{c+\sigma\kappa-\lambda\kappa}\right]^{\frac{1}{2n}} \quad (39)$$

and

$$B_1 = n\sqrt{\frac{(n+1)\left\{\begin{array}{l} \omega+\alpha\kappa-b\omega\kappa \\ +a\kappa^2+\gamma\kappa^3 \end{array}\right\}}{a-bv\omega+3\gamma\kappa}} \quad (40)$$

respectively.

Next, integrating the imaginary part equation (34), with integration constant zero, implies

$$\gamma g'' = \left\{\begin{array}{l} v+\alpha+2a\kappa \\ +b\omega(\omega+\kappa v)+3\gamma\kappa^2 \end{array}\right\} g - (\sigma-3\lambda-2v)\frac{g^{2n+1}}{2n+1} \quad (41)$$

Proceeding, after separation of variables, one recovers

$$\sqrt{\frac{\sigma-(2n+1)\lambda-2nv}{(n+1)(2n+1)\gamma}}(x-vt) = \int \frac{dg}{g\sqrt{N^2-g^{2n}}} \quad (42)$$

where

$$N^2 = \frac{(n+1)(2n+1)\left\{\begin{array}{l} v+\alpha+2a\kappa \\ +3\gamma\kappa^2-\omega b(\omega+\kappa v) \end{array}\right\}}{\sigma-(2n+1)\lambda-2nv} \quad (43)$$

This gives

$$g(x,t) = A_2 \operatorname{sech}^{\frac{1}{n}}[B_2(x-vt)] \quad (44)$$

where the amplitude A_2 and the inverse width B_2 of the soliton are

$$A_2 = \left[\frac{\begin{Bmatrix}(n+1)\\ \times(2n+1)\end{Bmatrix}\begin{Bmatrix}v+\alpha+2a\kappa\\ +3\gamma\kappa^2-\omega b(\omega+\kappa v)\end{Bmatrix}}{\sigma-(2n+1)\lambda-2n\nu}\right]^{\frac{1}{2n}} \tag{45}$$

$$B_2 = n\sqrt{\frac{(n+1)\begin{Bmatrix}v+\alpha+2a\kappa\\ +3\gamma\kappa^2-\omega b(\omega+\kappa v)\end{Bmatrix}}{\gamma}} \tag{46}$$

respectively. From (39) and (45), equating the two amplitudes A_1 and A_2 gives

$$\omega+\alpha\kappa-b\omega\kappa+a\kappa^2+\gamma\kappa^3 = \frac{(2n+1)\begin{Bmatrix}v+\alpha+2a\kappa+3\gamma\kappa^2\\ -b\omega(\omega+\kappa v)\end{Bmatrix}(c+\sigma\kappa-\lambda\kappa)}{\sigma-(2n+1)\lambda-2n\nu} \tag{47}$$

and then equating the two widths B_1 and B_2 reveals (25).

Finally, from (47) and (25), equating the two right hand sides leads to the constraint

$$\begin{aligned}&(2n+1)\gamma(c+\sigma\kappa-\lambda\kappa)\\ &=(\sigma-(2n+1)\lambda-2n\nu)(a-bv\omega+3\gamma\kappa)\end{aligned} \tag{48}$$

This relation between coefficients and parameters must be valid for the existence of bright soliton solution of SHE with STD in power law medium.

Thus, finally, the 1-soliton solution of (32) is

$$q(x,t) = A\,\text{sech}^{\frac{1}{n}}[B(x-vt)]e^{i(-\kappa x+\omega t+\theta)} \tag{49}$$

where the amplitude and the inverse width of the soliton is given by the pair (39)-(40) or (45)-(46). The wave number of the soliton are recoverable from (47) or (25). More constraint conditions are given by (28), (29) and (31). However, condition (30) now generalizes to

$$\begin{aligned}&(\sigma-(2n+1)\lambda-2n\nu)\\ &\times\left\{v+\alpha+2a\kappa+3\gamma\kappa^2-\omega b(\omega+\kappa v)\right\}>0\end{aligned} \tag{50}$$

that follows from (45). These conditions guarantee the existence of bright solitons for SHE with STD with power law nonlinearity. It needs to be noted that all results of Kerr law nonlinearity are recovered upon setting $n=1$ in this sub-section.

4. Conclusion

This paper obtains bright 1-soliton solution of SHE with STD in presence of several Hamiltonian type perturbation terms. Both Kerr and power law nonlinearities are considered. All the results of power law nonlinearity collapse to the case of Kerr law medium upon setting the power law nonlinearity parameter to unity. There are several constraint relations that naturally fell out from the algebraic structure of soliton parameters. The existence of bright solitons is guaranteed when these conditions hold.

The results of this paper are generalized version of a previously reported paper where SHE was studied without STD [1]. Moreover, it was only power law nonlinearity that was reported there. The results of this paper stand on a strong footing for pursuing further studies with SHE in presence of STD. It is clear that traveling wave hypothesis has limitations to the extraction of soliton solutions to SHE. This scheme only retrieves bright soliton solutions. Later, additional integration tools will display dark and singular soliton solutions to perturbed SHE with STD. Those results will be reported in future. Moreover, this study will be extended to birefringent fibers and DWDM system and the results of those researches will be published elsewhere.

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