Optical solitons for Kudryashov's model: undetermined coefficients with Jacobi's elliptic functions

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This paper recovers solitons solutions with Kudryashov's proposed law of refractive index having quadruple power-law of nonlinear refractive index. The method of undetermined coefficients is the integration algorithm where the starting hypotheses are Jacobi's elliptic functions. A full spectrum of soliton solutions are recovered when the modulus of ellipticity approaches unity.

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1. Introduction

One of the laws of refractive index, as proposed by N. Kudryashov, is with quadruple form of nonlinear refractive index [1, 2]. This law has been studied analytically all across by various authors and a wide range of research results have been reported. The soliton solutions have been recovered directly by the method of undetermined coefficients. Later, the model was also studied with nonlinear chromatic dispersion [1]. Various other studies have been done with regards to this proposed law of refractive index. The governing model is the nonlinear Schrödinger's equation (NLSE) with Kudryashov's law of nonlinear refractive index. The current paper will revisit the model and implement the method of undetermined coefficients, once again. This time the starting hypothesis would be Jacobi's elliptic function with modulus of ellipticity. Finally, the limiting value of this modulus of ellipticity, as it approaches unity, would yield soliton solutions. Thus, a full spectrum of solitons would emerge and this is exhibited in the manuscript with the necessary parameter constraints that are needed for the doubly periodic functions as well as the solitons to exist.

1.1. Governing model

KE, in the dimensionless form is given by [1, 2]

$$iq_t + aq_{xx} + \left(\frac{b_1}{|q|^{2n}} + \frac{b_2}{|q|^n} + b_3|q|^n + b_4|q|^{2n}\right)q = 0. \quad (1)$$

In (1), the linear temporal evolution is represented by the first term while the coefficient of chromatic dispersion is represented by a. The nonlinear terms b_j, for $1 \le j \le 4$, stem from the law of refractive index of an optical fiber and gives self–phase modulation effect to the model.

2. Mathematical analysis

This section will apply Jacobi's elliptic function ansatz approach to recover optical solitons to KE. For this, firstly, it is necessary in order to adopt the solution as

$$q(x,t) = P(x,t)e^{i\phi(x,t)} \quad (2)$$

where P stands for the amplitude portion, while the phase ϕ is structured as

$$\phi = -\kappa x + \omega t + \theta_0. \quad (3)$$

Here in (3), κ, ω and θ_0 are respectively the soliton frequency, its wave number and the phase constant. Plugging (2) into (1) and equating the real and imaginary parts respectively leads to

$$aP^{2n}\frac{\partial^2 P}{\partial x^2} + b_1 P + b_2 P^{n+1} + b_3 P^{3n+1} + b_4 P^{4n+1} - (\omega + a\kappa^2)P^{2n+1} = 0 \quad (4)$$

and

$$v = -2a\kappa \tag{5}$$

where v represents the speed of the soliton. Real portion given by (4) will now be examined by the use of four forms of Jacobi ansatz approach in subsequent subsections.

2.1. Bright solitons

The starting hypothesis for the first kind of Jacobi ansatz method is

$$P(x,t) = \frac{A}{(B+\mathrm{nc}\varsigma)^p} \tag{6}$$

where

$$\varsigma = \mu(x - vt). \tag{7}$$

Here in (6) and (7), A and B are respectively the amplitude and an external parameter, while μ is the inverse width and v is the soliton speed. The unknown exponent p will be identified later. It should be noted that when the modulus $k \to 1^-$, the solution (6) degenerates into bright soliton as [1]

$$P(x,t) = \frac{A}{(B+\cosh\varsigma)^p}. \tag{8}$$

Next, inserting (6) into (4) gives

$$\frac{b_1}{(B+\mathrm{nc}\varsigma)^{-2}} + \frac{b_2 A^n}{(B+\mathrm{nc}\varsigma)^{np-2}} - \frac{ap\mu^2(k^2-1)(p-1)A^{2n}}{(B+\mathrm{nc}\varsigma)^{2np-4}} + \frac{2ap\mu^2(k^2-1)(2p-1)A^{2n}B}{(B+\mathrm{nc}\varsigma)^{2np-3}} - \frac{[a(p^2\mu^2\{6B^2(k^2-1)-2k^2+1\}+\kappa^2)+\omega]A^{2n}}{(B+\mathrm{nc}\varsigma)^{2np-2}} + \frac{ap\mu^2(2p+1)\{2B^2(k^2-1)-2k^2+1\}A^{2n}B}{(B+\mathrm{nc}\varsigma)^{2np-1}} - \frac{ap\mu^2(p+1)(B^2-1)\{B^2(k^2-1)-k^2\}A^{2n}}{(B+\mathrm{nc}\varsigma)^{2np}} + \frac{b_3A^{3n}}{(B+\mathrm{nc}\varsigma)^{3np-2}} + \frac{b_4A^{4n}}{(B+\mathrm{nc}\varsigma)^{4np-2}} = 0. \tag{9}$$

Equating the exponents $2np$ and $4np-2$ in (9) yields

$$p = \frac{1}{n}. \tag{10}$$

Putting this value of p in (9), and collecting the coefficients of functions of the same exponent of $1/(B+\mathrm{nc}\varsigma)^j$ for $j = 0, \pm1, \pm2$, respectively, where each must be set to zero, leads to the system of algebraic equations given below

$$b_4n^2A^{2n} - a\mu^2(n+1)(B^2-1)\{B^2(k^2-1)-k^2\} = 0 \tag{11}$$

$$b_3n^2A^n + a\mu^2(n+2)B\{2B^2(k^2-1)-2k^2+1\} = 0 \tag{12}$$

$$n^2\omega + a[\mu^2\{6B^2(k^2-1)-2k^2+1\}+\kappa^2n^2] = 0 \tag{13}$$

$$b_2n^2 - 2a\mu^2(k^2-1)(n-2)A^nB = 0 \tag{14}$$

$$b_1n^2 + a\mu^2(k^2-1)(n-1)A^{2n} = 0. \tag{15}$$

Solving the above system yields

$$b_1 = -\frac{b_3^4(k^2-1)(n-1)(n+1)^3(B^2-1)^3\{B^2(k^2-1)-k^2\}^3}{b_4^3(n+2)^4B^4\{2B^2(k^2-1)-2k^2+1\}^4} \tag{16}$$

$$b_2 = -\frac{2b_3^3(k^2-1)(n-2)(n+1)^2(B^2-1)^2\{k^2-B^2(k^2-1)\}^2}{b_4^2(n+2)^3B^2\{2B^2(k^2-1)-2k^2+1\}^3} \tag{17}$$

$$A = \left[-\frac{b_3(n+1)(B^2-1)\{B^2(k^2-1)-k^2\}}{b_4(n+2)B\{2B^2(k^2-1)-2k^2+1\}}\right]^{\frac{1}{n}} \tag{18}$$

$$\mu = -\frac{nb_3}{(n+2)B\{2B^2(k^2-1)-2k^2+1\}}\sqrt{\frac{(n+1)(B^2-1)\{B^2(k^2-1)-k^2\}}{ab_4}} \tag{19}$$

$$\omega = -a\kappa^2 - \frac{b_3^2(n+1)(B^2-1)\{B^2(k^2-1)-k^2\}\{6B^2(k^2-1)-2k^2+1\}}{b_4(n+2)^2B^2\{2B^2(k^2-1)-2k^2+1\}^2}. \tag{20}$$

Here the solution is constrained by

$$ab_4(B^2-1)\{B^2(k^2-1)-k^2\} > 0 \tag{21}$$

and

$$b_4B\{2B^2(k^2-1)-2k^2+1\} \neq 0. \tag{22}$$

Thus, the solution in terms of elliptic function $nc\varsigma$ to the governing equation (1) is

$$q(x,t) = \frac{A}{\{B+\mathrm{nc}[\mu(x-vt)]\}^{\frac{1}{n}}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{23}$$

where the velocity v was given earlier by (5), the relation between the frequency of the soliton and its wave number is given by (20), while the amplitude and the width are respectively seen in (18) and (19). Finally, it can be concluded that the constraint conditions (21) and (22) must be satisfied for the soliton revealed to exist.

If $k \to 1^-$, the solution (23) becomes the following bright soliton:

$$q(x,t) = \frac{A}{\{B+\cosh[\mu(x-vt)]\}^{\frac{1}{n}}}e^{i(-\kappa x+\omega t+\theta_0)} \tag{24}$$

as long as the same constraint conditions (10), (14) and (15) given in Ref. [1] are provided. Also, the speed v, amplitude A, inverse width μ and wave number ω are respectively the same as Eqs. (5), (11), (12) and (13) given in Ref. [1]. As a results, when $k \to 1^-$, the solution (23) degenerates into the same bright soliton solution (16) in Ref. [1].

2.2. Dark solitons

For this form of Jacobi ansatz approach, a proper choice will be

$$P(x,t) = (A + B\mathrm{sn}\varsigma)^p \tag{25}$$

where ς is given as in Eq. (7). It should be mentioned that for $k \to 1^-$, the solution (25) degenerates into the following dark soliton solution [1]:

$$P(x,t) = (A + B\tanh\varsigma)^p. \tag{26}$$

Inserting (25) into (4) yields

$$\begin{aligned} & b_1 B^2 (A + B\mathrm{sn}\varsigma)^2 + b_2 B^2 (A + B\mathrm{sn}\varsigma)^{np+2} \\ & + ap\mu^2 (p-1)(A^2 - B^2)(k^2 A^2 - B^2)(A + B\mathrm{sn}\varsigma)^{2np} \\ & - ap\mu^2 (2p-1) A \{2k^2 A^2 - B^2 (k^2+1)\}(A + B\mathrm{sn}\varsigma)^{2np+1} \\ & - \begin{bmatrix} ap^2\mu^2\{k^2(B^2 - 6A^2) + B^2\} \\ + a\kappa^2 B^2 + \omega B^2 \end{bmatrix} (A + B\mathrm{sn}\varsigma)^{2np+2} \\ & - 2ak^2 p\mu^2 (2p+1) A (A + B\mathrm{sn}\varsigma)^{2np+3} \\ & + ak^2 p\mu^2 (p+1)(A + B\mathrm{sn}\varsigma)^{2np+4} \\ & + b_3 B^2 (A + B\mathrm{sn}\varsigma)^{3np+2} \\ & + b_4 B^2 (A + B\mathrm{sn}\varsigma)^{4np+2} = 0. \end{aligned} \tag{27}$$

Employing the balancing principle, one gets (10). Now from (27) noting that the functions $(A + B\mathrm{sn}\varsigma)^j$ for $0 \le j \le 4$ are linearly independent and so their coefficients must, individually vanish. This gives rise to

$$b_1 = -\frac{(n^2-1)\{4ab_4\mu^2(n+2)^2 + b_3^2 n^2 (n+1)\}\{4ab_4 k^2 \mu^2 (n+2)^2 + b_3^2 n^2 (n+1)\}}{16 b_4^3 n^4 (n+2)^4} \tag{28}$$

$$b_2 = -\frac{b_3 (n-2)(n+1)\{2ab_4\mu^2(k^2+1)(n+2)^2 + b_3^2 n^2 (n+1)\}}{4 b_4^2 n^2 (n+2)^3} \tag{29}$$

$$A = -\frac{b_3(n+1)}{2b_4(n+2)} \tag{30}$$

$$B = \frac{k\mu}{n}\sqrt{-\frac{a(n+1)}{b_4}} \tag{31}$$

$$\omega = -a\kappa^2 - \frac{a\mu^2(k^2+1)}{n^2} - \frac{3b_3^2(n+1)}{2b_4(n+2)^2} \tag{32}$$

provided

$$ab_4 < 0. \tag{33}$$

Thus, the solution in terms of elliptic function $sn\varsigma$ to the model (1) is

$$q(x,t) = \{A + B\mathrm{sn}[\mu(x - vt)]\}^{\frac{1}{n}} e^{i(-\kappa x + \omega t + \theta_0)} \tag{34}$$

where the soliton parameters A, B and ω are, respectively given by (30), (31) and (32) while the parameters b_1 and b_2 are presented as in Eqs. (28) and (29), and the constrain condition for the solution to exist is indicated by (33). Also, the velocity v was mentioned earlier in (5).

When $k \to 1^-$, from the solution (34) occurs dark soliton as

$$q(x,t) = \{A + B\tanh[\mu(x - vt)]\}^{\frac{1}{n}} e^{i(-\kappa x + \omega t + \theta_0)} \tag{35}$$

as long as the same constraint conditions (21) for $n = 1$ and (25) given in Ref. [1] are provided. Also, the speed v, wave number ω and amplitudes A and B are respectively the same as Eqs. (5), (22), (23) and (24) given in Ref. [1]. As a results, when $k \to 1^-$, the solution (34) degenerates into the same dark soliton solution (27) in Ref. [1].

2.3. Singular solitons (Form-I)

For this ansatz, the wave form hypothesis is

$$P(x,t) = \frac{A}{(B + \mathrm{sc}\varsigma)^p} \tag{36}$$

where ς is given by Eq. (7). Noting that when $k \to 1^-$, the solution (36) degenerates into the following singular soliton solution [1]:

$$P(x,t) = \frac{A}{(B + \sinh\varsigma)^p}. \tag{37}$$

The substitution of (36) into (4) leads to

$$\begin{aligned} & \frac{b_1}{(B + \mathrm{sc}\varsigma)^{-2}} + \frac{b_2 A^n}{(B + \mathrm{sc}\varsigma)^{np-2}} \\ & - \frac{ap\mu^2(k^2-1)(p-1)A^{2n}}{(B + \mathrm{sc}\varsigma)^{2np-4}} \\ & + \frac{2ap\mu^2(k^2-1)(2p-1)A^{2n}B}{(B + \mathrm{sc}\varsigma)^{2np-3}} \\ & - \frac{[a(\mu^2 p^2\{6B^2(k^2-1) + k^2 - 2\} + \kappa^2) + \omega]A^{2n}}{(B + \mathrm{sc}\varsigma)^{2np-2}} \\ & + \frac{ap\mu^2(2p+1)\{2B^2(k^2-1) + k^2 - 2\}A^{2n}B}{(B + \mathrm{sc}\varsigma)^{2np-1}} \\ & - \frac{ap\mu^2(p+1)\{B^2(k^2-1) - 1\}A^{2n}(B^2+1)}{(B + \mathrm{sc}\varsigma)^{2np}} \\ & + \frac{b_3 A^{3n}}{(B + \mathrm{sc}\varsigma)^{3np-2}} + \frac{b_4 A^{4n}}{(B + \mathrm{sc}\varsigma)^{4np-2}} = 0. \end{aligned} \tag{38}$$

From the balancing principle, one obtains the value of the parameter p as in (10). Now from (38) noting that the functions $1/(B + \mathrm{sc}\varsigma)^j$ for $0 \le j \le 4$ are linearly independent and therefore their coefficients must, individually vanish. This brings about

$$b_1 = -\frac{b_3^4(k^2-1)(n-1)(n+1)^3(B^2+1)^3\{B^2(k^2-1)-1\}^3}{b_4^3(n+2)^4 B^4\{2B^2(k^2-1)+k^2-2\}^4} \tag{39}$$

$$b_2 = -\frac{2b_3^3(k^2-1)(n-2)(n+1)^2(B^2+1)^2\{B^2(k^2-1)-1\}^2}{b_4^2(n+2)^3 B^2\{2B^2(k^2-1)+k^2-2\}^3} \tag{40}$$

$$A=\left[-\frac{b_3(n+1)(B^2+1)\{B^2(k^2-1)-1\}}{b_4(n+2)B\{2B^2(k^2-1)+k^2-2\}}\right]^{\frac{1}{n}} \quad (41)$$

$$\mu=\frac{nb_3}{B(n+2)\{2B^2(1-k^2)-k^2+2\}}\sqrt{\frac{(n+1)(B^2+1)\{B^2(k^2-1)-1\}}{ab_4}} \quad (42)$$

$$\omega=-a\kappa^2-\frac{b_3^2(n+1)(B^2+1)\{B^2(k^2-1)-1\}\{6B^2(k^2-1)+k^2-2\}}{b_4(n+2)^2B^2\{2B^2(k^2-1)+k^2-2\}^2}. \quad (43)$$

Here the solution is constrained by

$$ab_4\{B^2(k^2-1)-1\}>0 \quad (44)$$

and

$$b_4B\{2B^2(k^2-1)+k^2-2\}\neq 0. \quad (45)$$

Thus, the solution in terms of elliptic function $sc\,\varsigma$ to Eq. (1) is

$$q(x,t)=\frac{A}{\{B+\mathrm{sc}[\mu(x-vt)]\}^{\frac{1}{n}}}e^{i(-\kappa x+\omega t+\theta_0)} \quad (46)$$

where the parameters and constrains are given by from (39) to (45).

If $k\to 1^-$, the solution (46) becomes the following singular type-1 soliton solution:

$$q(x,t)=\frac{A}{\{B+\sinh[\mu(x-vt)]\}^{\frac{1}{n}}}e^{i(-\kappa x+\omega t+\theta_0)} \quad (47)$$

as long as the same constraint conditions (15) and (33) given in Ref. [1] are provided. Also, the speed v, values of parameters A and B, and wave number ω are respectively the same as Eqs. (5), (30), (31) and (32) given in Ref. [1]. As a results, when $k\to 1^-$, the solution (46) degenerates into the same singular type-1 soliton solution (34) in Ref. [1].

2.4. Singular solitons (Form-II)

For the last type of Jacobi ansatz, a proper choice would be

$$P(x,t)=(A+B\mathrm{ns}\varsigma)^p \quad (48)$$

where ς given by (7) is considered. It should be noted that when $k\to 1^-$, the solution (48) degenerates into the following the second type of singular soliton [1]:

$$P(x,t)=(A+B\coth\varsigma)^p. \quad (49)$$

Thus, substituting (48) into (4) yields

$$\begin{gathered}b_1B^2(A+B\mathrm{ns}\varsigma)^2+b_2B^2(A+B\mathrm{ns}\varsigma)^{np+2}\\ -ap\mu^2(p-1)(A^2-B^2)(A^2-k^2B^2)(A+B\mathrm{ns}\varsigma)^{2np}\\ +ap\mu^2(2p-3)A\{2A^2-B^2(k^2+1)\}(A+B\mathrm{ns}\varsigma)^{2np+1}\\ +\begin{bmatrix}ap\mu^2(p-2)\{B^2(k^2+1)-6A^2\}\\ -a\kappa^2B^2-\omega B^2\end{bmatrix}(A+B\mathrm{ns}\varsigma)^{2np+2}\\ -2ap\mu^2(2p+1)A(A+B\mathrm{ns}\varsigma)^{2np+3}\\ -ap\mu^2(p-3)(A+B\mathrm{ns}\varsigma)^{2np+4}\\ +b_3B^2(A+B\mathrm{ns}\varsigma)^{3np+2}\\ +b_4B^2(A+B\mathrm{ns}\varsigma)^{4np+2}=0.\end{gathered} \quad (50)$$

The balancing principle gives the same relation as in (10). Then, equating the coefficients of $(A+B\mathrm{ns}\varsigma)^j$ for $0\le j\le 4$ to zero, and solving the resulting systems yields

$$b_1=\frac{(n-1)(3n-1)\{4ab_4\mu^2(n+2)^2+b_3^2n^2(3n-1)\}\{4ab_4k^2\mu^2(n+2)^2+b_3^2n^2(3n-1)\}}{16b_4^3n^4(n+2)^4} \quad (51)$$

$$b_2=\frac{b_3\{9n(n-1)+2\}\{2ab_4\mu^2(k^2+1)(n+2)^2+b_3^2n^2(3n-1)\}}{4b_4^2n^2(n+2)^3} \quad (52)$$

$$A=\frac{b_3(1-3n)}{2b_4(n+2)} \quad (53)$$

$$B=\frac{\mu}{n}\sqrt{-\frac{a(3n-1)}{b_4}} \quad (54)$$

$$\omega=-a\kappa^2-\frac{a\mu^2(k^2+1)(2n-1)}{n^2}-\frac{3b_3^2\{n(6n-5)+1\}}{2b_4(n+2)^2} \quad (55)$$

provided

$$ab_4<0. \quad (56)$$

Thus, the solution in terms of elliptic function $ns\varsigma$ to KE is secured as

$$q(x,t)=\{A+B\mathrm{ns}[\mu(x-vt)]\}^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \quad (57)$$

where the parameters and constrain are demonstrated by from (51) to (56).

If $k\to 1^-$, the solution (57) becomes the following type-2 singular soliton:

$$q(x,t)=\{A+B\coth[\mu(x-vt)]\}^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta_0)} \quad (58)$$

as long as the same constraint conditions (21) for $n=1$ and (25) given in Ref. [1] are provided. Also, the speed v, wave number ω, and values of parameters A and B are respectively the same as Eqs. (5), (22), (23) and (24) given in Ref. [1]. As a results, when $k\to 1^-$, the solution (57) degenerates into the same type-2 singular soliton solution (37) in Ref. [1].

3. Conclusions

This paper recovered soliton solutions to the governing model that is the NLSE with Kudryashov's quadruple power–law of nonlinearity. The starting hypothesis being Jacobi's elliptic functions led to a wider variety of solutions and they are in terms of doubly periodic functions with the modulus of ellipticity. The limiting value of this modulus of ellipticity, when it approaches unity, yielded a full spectrum of soliton solutions. Therefore, these elliptic function solutions are being reported for the first time in this paper. Additionally, with these solutions, one can study cnoidal and snoidal waves that has not yet been explored for such a model. The analysis will be far and wide that would lead to a wide range of novel ideas. Later, further analysis would be conducted that aligns with previously reported results for a number of additional models [3–10]. This is just a tip of the iceberg!

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