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Optical solitons in nano-fibers with Kundu-Eckhaus equation by Lie symmetry analysis

SACHIN KUMAR[a], QIN ZHOU[b,*], ANJAN BISWAS[c,d], MILIVOJ BELIC[e]
[a]*Centre of Mathematics & Statistics, School of Basic and Applied Sciences, Central University of Punjab, City Campus, Mansa Road, Bathinda-151001, India*
[b]*School of Electronics and Information Engineering, Wuhan Donghu University, Wuhan, 430212, P.R. China*
[c]*Department of Mathematical Sciences, Delaware State University, Dover, DE 19901-2277, USA*
[d]*Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah-21589, Saudi Arabia*
[e]*Science Program, Texas A & M University at Qatar, PO Box 23874, Doha, Qatar*

This paper studies similarity solutions of Kundu-Eckhaus (KE) equation that models wave propagation in a dispersive medium such as in optical waveguide. Lie classical method is applied to obtain symmetries of KE equation and then using these symmetries reduction to ordinary differential equations (ODEs) is obtained. The corresponding exact solutions of KE equation are also presented.

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1. Introduction

Optical solitons in dispersive media is a very demanding topic of research in the field of nanophotonics [1-10]. There are a couple of models that are studied in this context. One of them is the popular Schrödinger-Hirota equation (SHE). There are several results that are reported in this context. This paper however will address another equation that is less poipular but is gradually gaining attention currently [3, 5, 6, 9, 10]. This is the Kundu-Eckhaus (KE) equation. Very recently dark and singular soliton solutions are reported for this model [3]. This paper will study KE equation from the point of view of Lie symmetry and group analysis. There are several forms of solution that are obtained in this paper, which will be of great asset in fiber optics community.

2. Governing equation

The KE equation is in the following form:

$$iq_t + aq_{xx} + b|q|^4 q + c(|q|^2)_x q = 0 \tag{1}$$

where $q = q(x,t)$ is a complex-valued function. This equation is a nonlinear Schrödinger type. The KE equation was presented by Calogero and Eckhaus [2] as an asymptotic multiscale reduction of certain classes of nonlinear partial differential equations.

3. Symmetry analysis

In this section, we will perform Lie symmetry analysis [1, 7, 8, 4] for equation (1). Firstly let us assume

$$q(x,t) = u(x,t) + iv(x,t) \tag{2}$$

Substituting (2) into (1), and separating real and imaginary parts, we have

$$u_t + av_{xx} + b(u^2+v^2)^2 v + 2cuvu_x + 2cv^2 v_x = 0 \tag{3}$$

$$-v_t + au_{xx} + b(u^2+v^2)^2 u + 2cuvv_x + 2cu^2 u_x = 0 \tag{4}$$

Let us consider one parameter Lie group of transformation

$$u^* \to u + \varepsilon\eta_1(x,t,u,v) \tag{5}$$

$$v^* \to v + \varepsilon\eta_2(x,t,u,v) \tag{6}$$

$$x^* \to x + \varepsilon\xi(x,t,u,v) \tag{7}$$

$$t^* \to t + \varepsilon\tau(x,t,u,v) \tag{8}$$

with small parameter $\varepsilon << 1$.

The associated vector field with the above group of transformations can be written as

$$V = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \eta_1(x,t,u,v)\frac{\partial}{\partial u} + \eta_2(x,t,u,v)\frac{\partial}{\partial v} \quad (9)$$

Applying the second prolongation $\mathrm{pr}^{(2)}V$ of V to Eqs. (3) and (4), we find that the coefficient functions ξ, τ, η_1 and η_2, must satisfy the invariance condition

$$\begin{aligned}&\eta_1^t + a\eta_2^{xx}\\ &+ b(4u^3v\eta_1 + u^4\eta_2 + 5v^4\eta_2 + 4uv^3\eta_1 + 6u^2v^2\eta_2)\\ &+ 2c(\eta_1 vu_x + uu_x\eta_2 + uv\eta_1^x)\\ &+ 2c(2v\eta_2 v_x + v^2\eta_2^x) = 0\end{aligned} \quad (10)$$

$$\begin{aligned}&-\eta_1^t + a\eta_1^{xx}\\ &+ b(4v^3u\eta_2 + v^4\eta_1 + 5v^4\eta_2 + 4vu^3\eta_2 + 6u^2v^2\eta_1)\\ &+ 2c(\eta_1 vv_x + uv_x\eta_2 + uv\eta_2^x)\\ &+ 2c(2u\eta_1 u_x + u^2\eta_1^x) = 0\end{aligned} \quad (11)$$

Substituting the infinitesimals η_1^t, η_2^t, η_1^{xx}, η_2^{xx} into equations (6), then using the equations (3) and (4), and equating the coefficients of the various monomials in the first, second and the other order partial derivatives with respect to x and various powers of u, we get the determining equations. Solving these determining equations, we get the following forms of the infinitesimals

$$\xi = C_2 + \frac{x}{2}C_4 + tC_5 + \frac{xt}{2}C_6 \quad (12)$$

$$\tau = C_1 + tC_4 + \frac{t^2}{2}C_6 \quad (13)$$

$$\eta_1 = -vC_3 - \frac{u}{4}C_4 - \frac{vx}{2a}C_5 - \frac{2uat + vx^2}{8}C_6 \quad (14)$$

$$\eta_2 = uC_3 - \frac{v}{4}C_4 + \frac{ux}{2a}C_5 - \frac{2vat - ux^2}{8}C_6 \quad (15)$$

where C_1, C_2, C_3, C_4, C_5 and C_6 are arbitrary constants.

Corresponding vector fields are

$$V_1 = \frac{\partial}{\partial t} \quad (16)$$

$$V_2 = \frac{\partial}{\partial x} \quad (17)$$

$$V_3 = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v} \quad (18)$$

$$V_4 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{u}{4}\frac{\partial}{\partial u} - \frac{v}{4}\frac{\partial}{\partial v} \quad (19)$$

$$V_5 = t\frac{\partial}{\partial x} - \frac{vx}{2a}\frac{\partial}{\partial u} + \frac{ux}{2a}\frac{\partial}{\partial v} \quad (20)$$

$$V_6 = \frac{xt}{2}\frac{\partial}{\partial x} + \frac{t^2}{2}\frac{\partial}{\partial t} - \left(\frac{2uat + vx^2}{8}\right)\frac{\partial}{\partial u} - \left(\frac{2vat - ux^2}{8}\right)\frac{\partial}{\partial v} \quad (21)$$

Let us consider following vector fields for the reduction of system of equations (3) and (4)

(1) $\mu V_1 + \lambda V_2 + V_3$

(2) V_4

(3) V_5

(4) V_6

where μ and λ are arbitrary constants.

For each case, one can get the reduction using characteristic equations:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta_1} = \frac{dv}{\eta_2} \quad (22)$$

4. Similarity reductions and exact solutions

4.1 Vector field $\mu V_1 + \lambda V_2 + V_3$

Corresponding similarity variables are as follows:

$$\xi = \mu x - \lambda t \quad (23)$$

$$q(x,t) = F(\xi)e^{i\left(\frac{x}{\lambda} + G(\xi)\right)} \quad (24)$$

where ξ is new independent variable and F is new dependent variable.

Substituting (23) and (24) in (1), we have following system of ODEs

$$(-\lambda^2 + 2a\mu)F' + a\mu^2\lambda(2F'G' + F^2G'') = 0 \quad (25)$$

$$
\begin{aligned}
&(\lambda^2 - 2a\lambda\mu - a\mu^2\lambda^2)FG' - aF + a\mu^2\lambda^2 F'' \\
&+ b\lambda^2 F^5 + 2c\lambda\mu^2 F^2 F' = 0
\end{aligned} \tag{26}
$$

where prime denotes derivatives with respect to ξ.

Solution of ODE system (25) and (26) is as follows

$$F = C_1 \tag{27}$$

$$G = \left(\frac{C_2}{a\mu^2\lambda C_1^2} + \frac{\lambda^2 - 2a\mu}{C_1}\right)\xi + C_3 \tag{28}$$

where

$$C_2 = -\frac{a\mu^2 C_1 \begin{pmatrix} b\lambda^2 C_1^5 - aC_1 + \lambda^5 - 4\lambda^3 a\mu \\ + 4\lambda a^2\mu^2 - \lambda^4 a\mu^2 + 2\lambda^2 a^2\mu^3 \end{pmatrix}}{-\lambda^2 + 2a\mu + a\lambda\mu^2} \tag{29}$$

and C_1, C_3 are arbitrary constants.

Corresponding solution of main system (1) is as follows

$$q(x,t) = C_1 e^{i\left(\frac{x}{\lambda} + \left(\frac{C_2}{a\mu^2\lambda C_{12}} + \frac{\lambda^2 - 2a\mu}{C_1}\right)(\mu x - \lambda t) + C_3\right)} \tag{30}$$

where C_2 is given by (29).

4.2 Vector field V_4

Corresponding similarity variables are

$$\sigma = \frac{x^2}{t} \tag{31}$$

$$u = t^{-\frac{1}{4}} H(\sigma) \tag{32}$$

$$v = t^{-\frac{1}{4}} G(\sigma) \tag{33}$$

where σ is new independent variable and F, G are new dependent variables.

Using the similarity variables (31)-(33) in (1), we obtain following system of ODEs

$$
\begin{aligned}
&- H - 4\sigma H' + 16a\sigma J'' + 8aJ' \\
&+ 4bJH^4 + 8bH^2 J^3 + 4bJ^5 \\
&+ 16c\sqrt{\sigma} HH'J + 16\sqrt{\sigma} J^2 J' = 0
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
&J + 4\sigma J' + 16a\sigma H'' + 8aH' \\
&+ 4bH^5 + 8bH^3 J^2 + 4bHJ^5 \\
&+ 16c\sqrt{\sigma} FGG' + 16c\sqrt{\sigma} H^2 H' = 0
\end{aligned} \tag{35}
$$

where the prime denotes derivatives with respect to σ.

4.3 Vector field V_5

Solving characteristic equation (22), we have following similarity variables

$$\varsigma = t \tag{36}$$

$$q = p(\varsigma) e^{i\left(\frac{x^2}{4at} + Q(\varsigma)\right)} \tag{37}$$

where ς is new independent variable and P, Q are new dependent variables.

Substituting the similarity variables (36) and (37) into (1), we have

$$2\varsigma P' + P = 0 \tag{38}$$

$$q' - bP^4 = 0 \tag{39}$$

where the prime denotes derivatives with respect to ς.

Solving (38) and (39), we obtain following general solution

$$P = \frac{C_1}{\sqrt{\varsigma}} \tag{40}$$

$$Q = -\frac{bC_1^4}{\sqrt{\varsigma}} + C_2 \tag{41}$$

where C_1 and C_2 are arbitrary constants.

Corresponding solution of main equation (1) is given by

$$q(x,t) = \frac{C_1}{\sqrt{t}} e^{i\left(\frac{x^2}{4at} - \frac{bC_1^4}{t} + C_2\right)} \tag{42}$$

4.4 Vector field V_6

Similarity variables are

$$\theta = \frac{x}{t} \tag{43}$$

$$q(x,t)=\frac{S(\theta)}{\sqrt{t}}e^{i(\frac{\pi}{2}+\frac{x^2}{4at}-T(\theta))} \tag{44}$$

where θ is new independent variable and S, T are new dependent variables.

Now using (21) in (1) and separating real and imaginary parts, we have

$$-aS''+aST'^2-bS^5-2cS^2S'=0 \tag{45}$$

$$2S'T'+ST''=0 \tag{46}$$

where the prime denotes derivatives with respect to θ.

We obtain following solutions of (22)

(1) $S=C_1$, $T=\frac{C_2}{C_1^2}\theta+C_3$

(2) $S=\frac{c\pm\sqrt{c^2-4ab-b}}{\theta}$, $T=C_1$

where C_1, C_2 and C_3 are arbitrary constants.

Corresponding solutions of main equation (1) are as follows

(1) $q(x,t)=\frac{C_1}{\sqrt{t}}e^{i(\frac{\pi}{2}+\frac{x^2}{4at}-\frac{C_2}{C_1^2}(\frac{x}{t})+C_3)}$

(2) $q(x,t)=\frac{\sqrt{t}(c\pm\sqrt{c^2-4ab-b})}{x}e^{i(\frac{\pi}{2}+\frac{x^2}{4at}-C_1)}$

5. Conclusion

This paper studied symmetry analysis for KE equation. The Lie classical method is utilized for obtaining the group infinitesimals. Using obtained infinitesimals, the KE equation is reduced to ODEs. Corresponding to the reduced ODEs, certain exact solutions are presented that are going to be very useful. The obtained solutions have also been verified by substituting them back into the original equation using Maple.

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References

[1] G. W. Bluman, S. C. Anco, Springer, New York (2002).
[2] F. Calogero, W. Eckhaus, Inv. Probl. **3**, 229 (1987)
[3] J. V. Guzman, M. F. Mahmood, D. Milovic, E. Zerrad, A. Biswas, D. Milovic, Optoelectron. Adv. Mat. **9**, (2015).
[4] S. Kumar, K. Singh, R. K. Gupta, Commun. Nonlinear Sci. Numer. Simul. **17**, 1529 (2012).
[5] S. De Lillo, G. Sanchini, Math. Comput. Modell. **50**, 255 (2005).
[6] I. Mercader, A. Alonso, O. Batiste, Eur. Phys. J. E **15**, 311 (2014).
[7] P. J. Olver, Applications of Lie Groups to Differential Equations, Springer-Verlag (1993).
[8] L. V. Ovsiannikov, Group Analysis of Differential Equations, Acad. Press, New York (1982).
[9] N. Taghizadeh, A. Neirameh, S. Shokooh, Trends in Applied Science Research. **7**, 476 (2012).
[10] H. Zhang, Commun. Nonlinear Sci. Numer. Simul. **14**, 3220 (2009).

*Corresponding author: biswas.anjan@gmail.com