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Optical solitons with spatially-dependent coefficients by Lie symmetry

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The optical solitons in media with space-modulated dispersion and non-Kerr law nonlinearity have been investigated analytically by employing the Lie group method. Lie symmetries and canonical transformations are obtained. Four laws of nonlinearity that are Kerr law, parabolic law, power law and dual-power law are considered.

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1. Introduction

The study of optical solitons is important in the area of telecommunications and electromagnetics [1]. Optical solitons, the localized electromagnetic waves that transmit in nonlinear media without changing their width, amplitude and transverse velocity, are the outcome of delicate equilibrium between the dispersion (or diffraction) and nonlinearity [2-5]. Just for these unique properties make optical solitons as the most ideal carriers of information, are widely applied to the long distance optical communications and ultra-fast signal processing systems. Optical soliton communications have many advantages that the traditional optical fiber communications don't have, such as the high information capacity, long transmission distance, high transmission rate, low bit error rate, good confidentiality and strong anti-interference ability and so on.

The dynamics of the propagation of optical solitons through optical fibers is ruled by the nonlinear Schrödinger Eq. (NLSE) [1-10]. When the space-modulated group velocity dispersion (GVD) and non-Kerr law nonlinearity are considered, the governing Eq. is given by

$$iu_t + a(x)u_{xx} + b(x)F(|u|^2)u = 0 \tag{1}$$

In Eq. (1), the first, second and third terms, respectively, are the temporal evolution, GVD and non-Kerr law nonlinear terms. Here $a(x)$ and $b(x)$ represent the coefficients of GVD and non-Kerr law nonlinearity that are the functions of spatial variable x. The real-valued algebraic function F in Eq. (1) represents the type of non-Kerr law nonlinearity. For space-modulated dual-power law nonlinearity (it describes the saturation of the nonlinear refractive index) [2,6], the expression for F has the form $F(|u|^2) = |u|^{2n} + \gamma(x)|u|^{4n}$, then Eq. (1) becomes

$$iu_t + a(x)u_{xx} + b(x)|u|^{2n}u + c(x)|u|^{4n}u = 0 \tag{2}$$

where $c(x) = b(x)\gamma(x)$. It needs to be noted that the dual-power law nonlinearity falls back to Kerr law nonlinearity (when $\gamma = 0$ and $n = 1$), parabolic law nonlinearity (when $n = 1$) and power law nonlinearity (when $\gamma = 0$).

The main work described in this paper is to construct exact optical soliton solutions to Eq. (2) by employing the Lie group method. As a consequence, Lie symmetries and canonical transformations are obtained, and explicit soliton solutions are found. Finally, Other laws of nonlinearity are discussed. They are Kerr law, parabolic law and power law.

2. Lie group analysis

Assume that Eq. (2) has stationary solutions in the form

$$u(x,t) = \varphi(x)\exp(-i\lambda t) \tag{3}$$

where $\varphi(x)$ represents the soliton amplitude that is a real function of spatial variable x to be determined later, while λ is a non-zero real constant. Substituting Eq. (3) into Eq. (2) yields

$$\varphi_{xx} + f(x)\varphi + g(x)\varphi^{2n+1} + h(x)\varphi^{4n+1} = 0 \qquad (4)$$

where $f(x) = \lambda/a(x)$,
$g(x) = b(x)/a(x)$, $h(x) = c(x)/a(x)$.

Lie group method is a very powerful method to study nonlinear Eqs. arising in the field of nonlinear science. In this section, we will perform Lie group method [11-14] to the second-order nonlinear variable-coefficients ordinary differential Eq. (ODE), i.e. Eq. (4).

If Eq. (4) is invariant under the one-parameter Lie group of point transformations

$$x^* = x + \varepsilon\xi(x,\varphi) + o(\varepsilon^2) \qquad (5)$$

$$u^* = u + \varepsilon\eta(x,\varphi) + o(\varepsilon^2) \qquad (6)$$

with infinitesimal generator

$$V = \xi(x,\varphi)\frac{\partial}{\partial x} + \eta(x,\varphi)\frac{\partial}{\partial \varphi} \qquad (7)$$

where the coefficient functions $\xi(x,\varphi)$ and $\eta(x,\varphi)$ are to be determined later. The vector field (7) is a generator of point symmetry of Eq. (4) if

$$pr^{(2)}V[\varphi_{xx} + f(x)\varphi + g(x)\varphi^{2n+1} + h(x)\varphi^{4n+1}]\Big|_{(4)} = 0 \qquad (8)$$

where $pr^{(2)}V$ represents the second prolongation and is defined by

$$\begin{aligned} pr^{(2)}V = \xi(x,\varphi)\frac{\partial}{\partial x} + \eta(x,\varphi)\frac{\partial}{\partial \varphi} + \\ + \eta^{x}(x,\varphi)\frac{\partial}{\partial \varphi_x} + \eta^{xx}(x,\varphi)\frac{\partial}{\partial \varphi_{xx}} \end{aligned} \qquad (9)$$

here $\eta^{x}(x,\varphi)$ and $\eta^{xx}(x,\varphi)$ are given by

$$\eta^{x} = D_x(\eta) - \varphi_x D_x(\xi) = \eta_x + (\eta_\varphi - \xi_x)\varphi_x - \varphi_x^2\xi_\varphi \qquad (10)$$

$$\begin{aligned} \eta^{xx} &= D_x(\eta^{x}) - \varphi_{xx}D_x(\xi) = \\ &= \eta_{xx} + (\eta_\varphi - 2\xi_x)\varphi_{xx} + (2\eta_{x\varphi} - \xi_{xx} - 3\xi_\varphi\varphi_{xx})\varphi_x + \\ &+ (\eta_{\varphi\varphi} - 2\xi_{x\varphi})\varphi_x^2 - \xi_{\varphi\varphi}\varphi_x^3 \end{aligned} \qquad (11)$$

where D_x denotes the total derivative operator and is defined by

$$D_x = \frac{\partial}{\partial x} + \varphi_x\frac{\partial}{\partial \varphi} + \varphi_{xx}\frac{\partial}{\partial \varphi_x} \qquad (12)$$

From Eqs. (8)-(12), one can obtain the determining equations for the symmetry group

$$\xi_{\varphi\varphi} = 0 \qquad (13)$$

$$\eta_{\varphi\varphi} - 2\xi_{x\varphi} = 0 \qquad (14)$$

$$2\eta_{x\varphi} - \xi_{xx} - 3\xi_\varphi\varphi_{xx} = 0 \qquad (15)$$

$$\begin{aligned} &(\eta_{xx} + \eta_\varphi\varphi_{xx} - 2\xi_x\varphi_{xx}) + \xi(f'\varphi + g'\varphi^{2n+1} + h'\varphi^{4n+1}) \\ &+ [f + (2n+1)g\varphi^{2n} + (4n+1)h\varphi^{4n}]\eta = 0 \end{aligned} \qquad (16)$$

Solving Eqs. (13)-(16), we get the only Lie point symmetry generator of Eq. (4)

$$V = p(x)\frac{\partial}{\partial x} + [\frac{1}{2}p'(x) + l]\varphi\frac{\partial}{\partial \varphi} \qquad (17)$$

with the constraint conditions

$$g(x) = g_0 p^{-(n+2)}(x)\exp[-2nl\int_0^x p^{-1}(\varsigma)d\varsigma] \qquad (18)$$

$$h(x) = h_0 p^{-2(n+1)}(x)\exp[-4nl\int_0^x p^{-1}(\varsigma)d\varsigma] \qquad (19)$$

$$p'''(x) + 4f(x)p'(x) + 2f'(x)p(x) = 0 \qquad (20)$$

where l, g_0, and h_0 are the arbitrary real constants.

According to the invariance of the energy and translational invariance, we get the canonical transformation

$$U(\theta) = p^{-1/2}(x)\exp(-l\theta)\varphi(x) \qquad (21)$$

where $\theta = \int_0^x p^{-1}(\varsigma)d\varsigma$

Under the transformation (21), Eq. (4) reduces to a second-order constant-coefficients ODE reads

$$U_{\theta\theta} + 2lU_\theta + g_0U^{2n+1} + h_0U^{4n+1} = k_0U \qquad (23)$$

where $k_0 = \frac{1}{4}p'^2 - \frac{1}{2}pp'' - l^2 - fp^2$ is a constant due to $dk_0/dx = 0$ from Eq. (20).

It needs to be noted that one can get $p(x)$ by solving Eq. (20) when GVD coefficient is known, and then obtain the Lie point symmetry generator (17) and canonical transformation (21). Therefore it is our primary task now to seek analytical solutions to Eq. (23), for the sake of simplicity, we take $l = 0$ in our calculations.

3. Results and discussion

Integrating Eq. (23) once and choosing the integration constant to be zero, then making some simple calculations yields

$$\sqrt{k_0}\theta = \int \frac{dU}{U\sqrt{1-\frac{g_0}{k_0(n+1)}U^{2n}-\frac{h_0}{k_0(2n+1)}U^{4n}}} \tag{24}$$

Solving Eq. (24), one get

$$U(\theta)=\left\{\frac{4}{\left[\frac{g_0^2}{k_0^2(n+1)^2}+\frac{4h_0}{k_0(2n+1)}\right]e^{2n\sqrt{k_0}\theta}+e^{-2n\sqrt{k_0}\theta}+\frac{2g_0}{k_0(n+1)}}\right\}^{1/2n} \tag{25}$$

Now taking $(2n+1)(n+1)^2k_0^2-4(n+1)^2h_0k_0-(2n+1)g_0^2=0$ yields

$$k_0=\frac{2(n+1)h_0\pm\sqrt{4(n+1)^2h_0^2+(2n+1)^2g_0^2}}{(n+1)(2n+1)} \tag{26}$$

Thus, Eq. (25) gives the wave profile

$$U(\theta)=\left\{\frac{2}{\cosh(2n\sqrt{k_0}\theta)+\frac{g_0}{k_0(n+1)}}\right\}^{1/2n} \tag{27}$$

Finally, explicit 1-soliton solutions to Eq. (1) with dual-power law nonlinearity are got by substituting Eqs. (27) and (21) into Eq. (3).

3.1 Kerr law nonlinearity

The Kerr law nonlinearity, also known as the cubic nonlinearity, arises in a light pulse propagating in an optical fiber that faces nonlinear responses from non-harmonic motion of electrons bound in molecules [2, 6]. If $c=0$ (i.e. $\gamma=0$) and $n=1$, the dual-power law nonlinearity falls back to the Kerr law nonlinearity, then the governing Eq. of optical solitons through space-modulated optical fibers with Kerr law nonlinearity is given by

$$iu_t+a(x)u_{xx}+b(x)|u|^2u=0 \tag{28}$$

Based on the same ideas as in Section 2, we can get the following results: the Lie point symmetry generator of Eq. (28) is Eq. (17); the constraint conditions are Eq. (18) with $n=1$ and Eq. (20); the canonical transformation is also the Eq. (21).

Now we just need to solve the following Eq.

$$\sqrt{k_0}\theta = \int \frac{dU}{U\sqrt{1-\frac{g_0}{2k_0}U^2}} \tag{29}$$

Solving Eq. (29), one gets a bright soliton (bell soliton) profile

$$\mathrm{U}(\theta)=\sqrt{\frac{2k_0}{g_0}}\,\mathrm{sech}(\sqrt{k_0}\theta) \tag{30}$$

and a singular soliton profile

$$\mathrm{U}(\theta)=\sqrt{-\frac{2k_0}{g_0}}\,\mathrm{csch}(\sqrt{k_0}\theta) \tag{31}$$

Eq. (30) imposes the constraints $k_0>0$ and $g_0>0$, while Eq. (31) imposes the constraints $k_0>0$ and $g_0<0$.

Finally, explicit 1-soliton solutions to Eq. (1) with Kerr law nonlinearity (i.e. Eq. (28)) are got by substituting Eqs. (30), (31) and (21) into Eq. (3).

Here, we must emphasize that there exist more solutions, this is because, in this case, Eq. (23) reduces to the famous Jacobian elliptic Eq. reads

$$U_{\theta\theta}+g_0U^3=k_0U \tag{32}$$

and all solutions of which are given in Ref. [15].

3.2. Parabolic law nonlinearity

The parabolic law nonlinearity, also known as the cubic-quintic nonlinearity, originates from the nonlinear interaction between Langmuir waves and electrons [2, 6]. If $n=1$, the dual-power law nonlinearity reduces to the parabolic law nonlinearity, then the governing Eq. of optical solitons through space-modulated optical fibers with parabolic law nonlinearity is given by

$$iu_t+a(x)u_{xx}+b(x)|u|^2u+c(x)|u|^4u=0 \tag{33}$$

Based on the same ideas as in Section 2, we can get the following results: the Lie point symmetry generator of Eq. (33) is Eq. (17); the constraint conditions are Eqs. (18)-(19) with $n=1$ and Eq. (20); the canonical transformation is also the Eq. (21).

Now we just need to solve the following Eq.

$$\sqrt{k_0}\theta = \int \frac{dU}{U\sqrt{1-\frac{g_0}{2k_0}U^2-\frac{h_0}{3k_0}U^4}} \tag{34}$$

Solving Eq. (34), one get

$$U(\theta)=\frac{2}{\sqrt{\left(\frac{g_0^2}{4k_0^2}+\frac{4h_0}{3k_0}\right)\exp(2\sqrt{k_0}\theta)+\exp(-2\sqrt{k_0}\theta)+\frac{g_0}{k_0}}} \tag{35}$$

Now taking $3g_0^2+16h_0k_0=12k_0^2$ yields

$$k_0=\frac{4h_0\pm\sqrt{16h_0^2-9g_0^2}}{6} \tag{36}$$

Thus, Eq. (35) gives the wave profile

$$U(\theta)=\frac{2}{\sqrt{2\cosh(2\sqrt{k_0}\theta)+\frac{g_0}{k_0}}} \tag{37}$$

Finally, explicit 1-soliton solutions to Eq. (1) with parabolic law nonlinearity (i.e. Eq. (33)) are got by substituting Eqs. (35) and (21) into Eq. (3).

Here, we must also emphasize that there exist more solutions; this is because, in this case, Eq. (23) reduces to the famous U^6 model reads

$$U_{\theta\theta}+g_0U^3+h_0U^5=k_0U \tag{38}$$

and all solutions (constructed by using the Jacobian elliptic Eq. expansion method) of which are given in Ref. [15].

3.3. Power law nonlinearity

The power law nonlinearity arises in various materials [2, 6]. If $c=0$ (i.e. $\gamma=0$), the dual-power law nonlinearity reduces to power law nonlinearity, then the governing Eq. of optical solitons through space-modulated optical fibers with power law nonlinearity is given by

$$iu_t+a(x)u_{xx}+b(x)|u|^{2n}u=0 \tag{39}$$

Based on the same ideas as in Section 2, we can get the following results: the Lie point symmetry generator of Eq. (39) is Eq. (17); the constraint conditions are Eqs. (18) and (20); the canonical transformation is also the Eq. (21).

Now we just need to solve the following Eq.

$$\sqrt{k_0}\theta=\int\frac{dU}{U\sqrt{1-\frac{g_0}{k_0(n+1)}U^{2n}}} \tag{40}$$

Solving Eq. (40), one gets the wave profile

$$\mathrm{U}(\theta)=\left[\frac{k_0(n+1)}{g_0}\right]^{1/2n}\operatorname{sech}^{1/n}[n\sqrt{k_0}\theta] \tag{41}$$

Finally, explicit bright 1-soliton solution to Eq. (1) with power law nonlinearity (i.e. Eq. (39)) is got by substituting Eqs. (41) and (21) into Eq. (3).

4. Conclusions

Exact optical solitons to the NLSE with space-dependent dispersion and four types of non-Kerr law nonlinearity are constructed with the aid of the Lie group method. These nonlinearities are the Kerr law, parabolic law, power law and dual-power law. The Lie point symmetry generator and canonical transformation to the second-order nonlinear variable-coefficients ODE (i.e. Eq. (4)) are got.

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