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Solitons in optical metamaterials by F-expansion scheme

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We demonstrate the existence of solitons in special optical metamaterials with Kerr law nonlinearity. The F-expansion scheme is utilized to obtain solitons and traveling wave solutions to the governing wave evolution model. The existence of such solutions requires certain constraint conditions to be satisfied.

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1. Introduction

Soliton pulse propagation properties in complex materials with simultaneous negative real dielectric permittivity and magnetic permeability, also known as double negative (DNG) materials, have attracted much attention in recent research [1-20]. These types of materials are not found in nature, but rather need to be fabricated through material processed engineering. Therefore these materials are called metamaterials. The novelty of these engineered materials with their possible applications to support short duration optical soliton pulses is investigated in this paper.

Recently reported metamaterials in optical regions have shown promise to make an effective optical waveguide, according to Shalaev [15]. Theoretical study of nonlinear optical pulse propagation in metamaterial waveguides has been reported recently by some coauthors of the present paper [8]. Optical waveguide can be implemented by using slab structure where the core is regular positive-indexed material and claddings are DNG materials. As a realistic implementation of optical waveguide, a photonic crystal with gold nanoparticle in the crystal hole also shows negative refractive properties. Simulation demonstrates both forward and backward propagating waves in optical waveguides implemented in photonic crystal metamaterials that support soliton wave propagation. The dispersion and loss characteristics of the waveguide contribute a pivotal role in wave propagation properties. Presence of backward wave that is observed in the transmission simulation, demonstrates negative refraction [8].

This paper will carry out the integration of the model equation that describes the propagation of solitons and other waves through these metamaterial waveguides. The modified F-expansion scheme is employed to carry out the integration of generalized nonlinear Schrödinger equation (NLSE) with additional terms that account for the metamaterials. Solitons and traveling wave solutions will be retrieved from this model by the application of the F-expansion algorithm.

2. Overview of the modified F-expansion scheme

In this section, the basic algorithm of modified F-expansion method is introduced. The protocol is as follows:

Step-1: Consider a given nonlinear evolution partial differential equation with independent variables $x = x(t, x_1, x_2, \ldots, x_l)$ and dependent variable $u(x)$ as

$$P(u, u_t, u_{x1}, u_{x2}, \ldots, u_{xl}, u_{tt}, u_{x1x2}, u_{x1t}, u_{x1x1}, \ldots) = 0, \qquad (1)$$

in which the traveling wave variable transformation

$$u(t, x_1, x_2, \ldots, x_l) = U(\xi),\ \xi = k_1 x_1 + k_2 x_2 + \ldots + k_l x_l - vt \qquad (2)$$

is utilized, to explore the existence of traveling wave and solitary solutions. Here $k_1, k_2, \ldots, k_l$ and v are constants to be determined. Now, if one inserts (2) into (1), Eq. (1) is reduced to the nonlinear ordinary differential equation (ODE) as follows:

$$Q(U, U', U'', \ldots) = 0 \qquad (3)$$

Step-2: Suppose that $U(\xi)$ can be expressed as

$$U(\xi)=\sum_{i=-m}^{m}a_i F^i(\xi) \tag{4}$$

where a_i are constants to be determined, $a_m \neq 0$ and $F(\xi)$ satisfies Riccati equation of the form

$$F'(\xi)=A+BF(\xi)+CF^2(\xi) \tag{5}$$

where *A*, *B* and *C* are unknown constants, $C \neq 0$ and *m* is an integer number that can be determined by considering the homogenous balance between the highest order nonlinear term(s) and the highest order partial derivative in Eq. (3).

Step-3: Substitute (4) to ODE (3) and collect all terms with the same order of *F* together; then the left hand side of (3) is converted into a finite series in $F^p(\xi)\,(p=-m,\ldots,m)$. Equating each coefficient of this polynomial to zero, by using Maple one obtains a set of algebraic equations for a_i, k_j and *v* for $i=-m,\ldots,m$ and $j=1,2,\ldots,l$.

Step-4: Solve the system of algebraic equations for a_i, k_j and *v* that can be expressed by *A*, *B* and *C* (or the coefficients of ODE (3)). Substituting these results into (4), one can obtain the general form of traveling wave solutions to Eq. (3).

Step-5: With the help of Appendix, from the general form of traveling wave solutions, we can obtain various soliton-like solutions, trigonometric function solutions and rational solutions of Eq. (3).

3. Application to optical metamaterials

Consider the generalized one-dimensional NLSE, which in the dimensionless form is given by [1-4, 15]

$$\begin{aligned} &iq_t+aq_{xx}+b|q|^2q=i\alpha q_x+i\lambda(|q|^2q)_x+i\nu(|q|^2)_xq+\\ &+\theta_1(|q|^2q)_{xx}+\theta_2|q|^2q_{xx}+\theta_3q^2q^*_{xx} \end{aligned} \tag{6}$$

In equation (6), the first term represents temporal evolution, the coefficient of *a* is the group velocity dispersion while *b* is the coefficient of Kerr law nonlinearity. On the right hand side, various additional terms are placed that describe different physical phenomena of interest here; α is the coefficient of inter-modal dispersion, λ is the coefficient of self-steepening and ν is the nonlinear dispersion. Finally, θ_j (*j* = 1, 2, 3), where θ_j are real-valued constants, account for specific metamaterials that were introduced earlier and reported in [19]. That reference dealt with controllable Raman soliton self-frequency shift in metamaterials with nonlinear electric polarization.

Later, the integrability aspects of equation (6) were studied in [1-4]. This model was also studied in details by additional authors [12, 14, 16, 17]. More recently, soliton propagation in negative-indexed materials with self-steepening effect was studied in details in [11]. In that paper, the authors applied Darboux transform to solve the governing equation and thus reported bright 1-soliton solutions in 2014 [11]. The effect of self-steepening was also studied earlier by Zhou et. al. in 2012, where sub-ODE approach was employed [20].

The present paper will apply F-expansion scheme to solve the governing equation (6). In order to solve (6), we use the complex ansatz solution

$$q(x,t)=U(\xi)e^{i\Phi},\ \xi=\beta(x-vt) \tag{7}$$

where $U(\xi)$ represents the shape of the soliton and *v* is the velocity of the wave. Since $q(x,t)$ is a complex-valued wave profile, there exists a phase component that is given by

$$\Phi=-kx+\omega t+\theta \tag{8}$$

where *k* is the soliton wave number while ω is the frequency, and θ is the phase constant. Now Eq. (6), with the hypothesis (7), splits into real

$$\begin{aligned} &a\beta^2U''-(\omega+ak^2+\alpha k)U+(b-\lambda k+k^2(\theta_1+\theta_2+\theta_3))U^3-\\ &-(3\beta^2\theta_1+\beta^2\theta_2+\beta^2\theta_3)U^2U''-6\beta^2\theta_1U(U')^2=0 \end{aligned} \tag{9}$$

and imaginary part

$$\begin{aligned} &(\beta v+2a\beta k+\alpha\beta)U'+\\ &+(3\lambda\beta+2\beta v-6\beta k\theta_1-2\beta k\theta_2+2\beta k\theta_3)U^2U'=0 \end{aligned} \tag{10}$$

From (10), it is possible to observe that U' and U^2U' are linearly independent functions and hence their coefficients must be zero, which implies the constraint conditions on the model coefficients and parameters

$$\begin{aligned} &v=-\alpha-2a\beta\\ &-3\lambda-2v+2\beta(3\theta_1+\theta_2-\theta_3)=0 \end{aligned} \tag{11}$$

By balancing U^3 with *U”* in Eq. (9), we get *m* = 1. Thus, we may choose

$$U(\xi)=a_0+a_{-1}F^{-1}(\xi)+a_1F(\xi) \tag{12}$$

Substituting (12) into (9) and using (5), the left hand side of Eq. (9) can be converted into a finite series in *F*; equating each coefficient in this series to zero yields a system of algebraic equations for a_0, a_{-1}, a_1, k, ω, θ_1, θ_2 and θ_3. Solving the algebraic equations using Maple, we find the following solutions of a_0, a_{-1}, a_1, k, ω, θ_1, θ_2 and θ_3.

Case-1: When *A* = 0, we have the following solutions:

$$a_0 = a_0,\ a_{-1} = 0,\ a_1 = \frac{2a_0C}{B},\ k = k$$
$$\omega = \frac{-1}{(10k^2 - 3\beta^2B^2)B^2}(-3\beta^2B^4k\lambda a_0^2 + 2k^2\beta^2B^4a + $$
$$+3\beta^2B^4a_0^2b - 3\beta^2B^4k\alpha + 10k^3\alpha B^2 + 10ak^4B^2)$$
$$\theta_1 = 0 \tag{13}$$
$$\theta_2 = \frac{-1}{(10k^2 - 3\beta^2B^2)a_0^2}(-12k\lambda a_0^2 + 6\beta^2B^2a +$$
$$+10\theta 3k^2a_0^2 + 12a_0^2b - 3\theta_3\beta^2B^2a_0^2)$$
$$\theta_3 = \theta_3$$

and

$$a_0 = \frac{\sqrt{5}\sqrt{(\sqrt{30B^2}\beta\lambda - 10b)a}B\beta}{\sqrt{30B^2}\beta\lambda - 10b},$$
$$a_{-1} = 0,\ a_1 = \frac{2\sqrt{5}\sqrt{(\sqrt{30B^2}\beta\lambda - 10b)aC\beta}}{\sqrt{30B^2}\beta\lambda - 10b},$$
$$\omega = \frac{-\beta}{10(\sqrt{30B^2}\beta\lambda - 10b)}(8a\beta^2B^2\lambda\sqrt{30B^2} - \tag{14}$$
$$-10b\alpha\sqrt{30B^2} - 80a\beta B^2b + 75a\beta^3B^4\theta_1 + 30\beta B^2\lambda\alpha)$$
$$k = \frac{1}{10}\sqrt{30B^2}\beta,$$
$$\theta_1 = \theta_1,\ \theta_2 = -6\theta_1 - \theta_3,\ \theta_3 = \theta_3$$

Case-2: When $B = 0$, we have

$$a_0 = 0,\quad a_{-1} = 0,$$
$$a_1 = \frac{2\sqrt{5}\sqrt{(2\sqrt{-30CA}\beta\lambda - 10b)aC\beta}}{2\sqrt{-30CA}\beta\lambda - 10b},$$
$$\omega = \frac{-\beta}{10(2\sqrt{-30CA}\beta\lambda - 10b)}(-64a\beta^2CA\lambda\sqrt{-30CA} - \tag{15}$$
$$-20b\alpha\sqrt{-30CA} + 320a\beta CAb + 1200a\beta^3A^2C^2\theta_1 -$$
$$-120\beta CA\lambda\alpha),\qquad k = \frac{1}{5}\sqrt{-30CA}\beta,$$
$$\theta_1 = \theta_1,\ \theta_2 = -6\theta_1 - \theta_3,\ \theta_3 = \theta_3$$

Case-3: When A=B = 0, we have

$$a_0 = 0,\ \ a_{-1} = 0,\ a_1 = -\frac{1}{5}\frac{\sqrt{5}\sqrt{-10ba}\,C\beta}{b}, \tag{16}$$
$$\omega = 0,\ k = 0,\ \theta_1 = \theta_1,\ \theta_2 = -6\theta_1 - \theta_3,\ \theta_3 = \theta_3$$

Substituting these solutions into (12) and using Appendix, we can obtain different soliton-like solutions, trigonometric function solutions and rational solutions to Eq. (9) (where we left the same type solutions out). They are listed in the following subsection.

3. 1 Solitonic, periodic, singular periodic and plane wave solutions

(I) When A = 0, B = 1, C = -1; from Appendix, then $F(\xi) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2}\xi\right)$. By Case 1, we have soliton-like solutions of Eq. (9) as

$$U_1(\xi) = -a_0\tanh\left(\frac{1}{2}\xi\right),$$
$$U_2(\xi) = -\frac{\sqrt{5}\sqrt{(\sqrt{30}\beta\lambda - 10b)a}\beta\tanh\left(\frac{1}{2}\xi\right)}{\sqrt{30}\beta\lambda - 10b}.$$

(II) When A = 0, B = -1, C = 1; from Appendix, then $F(\xi) = \frac{1}{2} - \frac{1}{2}\coth\left(\frac{1}{2}\xi\right)$. By Case 1, we have soliton-like solutions of Eq. (9) as

$$U_3(\xi) = a_0\coth\left(\frac{1}{2}\xi\right),$$
$$U_4(\xi) = -\frac{\sqrt{5}\sqrt{(\sqrt{30}\beta\lambda - 10b)a}\beta\coth\left(\frac{1}{2}\xi\right)}{\sqrt{30}\beta\lambda - 10b}.$$

(III) When $A = \frac{1}{2}$, B = 0, $C = -\frac{1}{2}$; $F(\xi) = \coth(\xi) \pm \operatorname{csch}(\xi)$ or $F(\xi) = \tanh(\xi) \pm i\operatorname{sech}(\xi)$. Thus

$$U_5(\xi) = -\frac{\sqrt{5}\sqrt{(\sqrt{30}\beta\lambda - 10b)a}\beta(\cosh(\xi) + 1)}{(\sqrt{30}\beta\lambda - 10b)\sinh(\xi)}$$
$$U_6(\xi) = -\frac{\sqrt{5}\sqrt{(\sqrt{30}\beta\lambda - 10b)a}\beta(\cosh(\xi) - 1)}{(\sqrt{30}\beta\lambda - 10b)\sinh(\xi)}$$
$$U_7(\xi) = -\frac{\sqrt{5}\sqrt{(\sqrt{30}\beta\lambda - 10b)a}\beta(\sinh(\xi) + i)}{(\sqrt{30}\beta\lambda - 10b)\cosh(\xi)}$$
$$U_8(\xi) = -\frac{\sqrt{5}\sqrt{(\sqrt{30}\beta\lambda - 10b)a}\beta(\sinh(\xi) - i)}{(\sqrt{30}\beta\lambda - 10b)\cosh(\xi)}$$

(IV) When A =1, B = 0, C = -1; $F(\xi) = \tanh(\xi)$ or $\coth h(\xi)$. By Case 2, we have soliton-like solutions of Eq. (9)

$$U_9(\xi) = -\frac{\sqrt{10}\sqrt{(\sqrt{30}\beta\lambda - 5b)a\beta}\tanh(\xi)}{\sqrt{30}\beta\lambda - 5b}$$

$$U_{10}(\xi) = -\frac{\sqrt{10}\sqrt{(\sqrt{30}\beta\lambda - 5b)a\beta}\coth(\xi)}{\sqrt{30}\beta\lambda - 5b}$$

(V) When $A = \frac{1}{2}$, $B = 0$, $C = \frac{1}{2}$; $F(\xi) = \sec(\xi) + \tan(\xi)$ or $\csc(\xi) - \cot(\xi)$. By Case 2, we have trigonometric function solutions of Eq. (9)

$$U_{11}(\xi) = \frac{\sqrt{5}\sqrt{(i\sqrt{30}\beta\lambda - 10b)a\beta}(\sin(\xi) + 1)}{(i\sqrt{30}\beta\lambda - 10b)\cos(\xi)}$$

$$U_{12}(\xi) = -\frac{\sqrt{5}\sqrt{(i\sqrt{30}\beta\lambda - 10b)a\beta}(\cos(\xi) - 1)}{(i\sqrt{30}\beta\lambda - 10b)\sin(\xi)}$$

(VI) When $A = -\frac{1}{2}$, $B = 0$, $C = -\frac{1}{2}$; $F(\xi) = \sec(\xi) - \tan(\xi)$ or $\csc(\xi) + \cot(\xi)$.

$$U_{13}(\xi) = -\frac{\sqrt{5}\sqrt{(i\sqrt{30}\beta\lambda - 10b)a\beta}(\cos(\xi) + 1)}{(i\sqrt{30}\beta\lambda - 10b)\sin(\xi)}$$

$$U_{14}(\xi) = \frac{\sqrt{5}\sqrt{(i\sqrt{30}\beta\lambda - 10b)a\beta}(\sin(\xi) - 1)}{(i\sqrt{30}\beta\lambda - 10b)\cos(\xi)}$$

(VII) When $A = 1$, $B = 0$, $C = 1$; from Appendix, then $F(\xi) = \tan(\xi)$. By Case 2, we have trigonometric function solutions of Eq. (9) as

$$U_{15}(\xi) = \frac{\sqrt{10}\sqrt{(i\sqrt{30}\beta\lambda - 5b)a\beta}\tan(\xi)}{i\sqrt{30}\beta\lambda - 5b}$$

(VIII) When $A = -1$, $B = 0$, $C = -1$; from Appendix, then $F(\xi) = \cot(\xi)$. By Case 2, we have trigonometric function solutions of Eq. (9)

$$U_{16}(\xi) = -\frac{\sqrt{10}\sqrt{(i\sqrt{30}\beta\lambda - 5b)a\beta}\cot(\xi)}{i\sqrt{30}\beta\lambda - 5b}$$

(XI) When $A = 0$, $B = 0$, $C \neq 0$; from Appendix, then $F(\xi) = -\frac{1}{C\xi + \lambda}$. By Case 3, we have rational solutions of Eq. (9)

$$U_{17}(\xi) = -\frac{\sqrt{-2baC\beta}}{b(C\xi + \lambda)}.$$

where C and λ are arbitrary constants and $C \neq 0$.

Thus, in all cases, the solutions to the generalized NLSE are written as

$$q(x,t) = U_j(\xi)e^{i\Phi}$$

for $1 \leq j \leq 17$.

4. Conclusions

This paper successfully applied the modified F-expansion scheme to solve the model evolution equation that governs the dynamics of soliton and wave propagation through special optical metamaterials. It is interesting to note that the solutions obtained by this integration scheme can be dark solitons, singular solitons, plane waves and singular periodic solutions. It is, however, only Kerr law nonlinearity that is taken into consideration. In future, other laws of nonlinearity will be studied. These include the power law, parabolic law, dual-power law, log law and others. The results of these nonlinear media will be reported elsewhere.

Appendix

Relations between A, B, C and the corresponding $F(\xi)$ in Riccati equation:

A	B	C	F
0	1	-1	$\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2}\xi\right)$
0	-1	1	$\frac{1}{2} - \frac{1}{2}\coth\left(\frac{1}{2}\xi\right)$
$\frac{1}{2}$	0	$-\frac{1}{2}$	$\coth(\xi) \pm \operatorname{csch}(\xi)$ $\tanh(\xi) \pm i\operatorname{sech}(\xi)$
1	0	-1	$\tanh(\xi)$, $\coth(\xi)$
$\frac{1}{2}$	0	$\frac{1}{2}$	$\sec(\xi) + \tan(\xi)$, $\csc(\xi) - \cot(\xi)$
$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\sec(\xi) - \tan(\xi)$, $\csc(\xi) + \cot(\xi)$
1(-1)	0	1(-1)	$\tan(\xi)$ ($\cot(\xi)$)
0	0	$\neq 0$	$-\frac{1}{C\xi + \lambda}$ (λ is arbitrary constant)

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