Some efficient derivative-free iterative methods for solving nonlinear equations

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In this paper, we suggest and analyze some new iterative methods for solving nonlinear equations. By using suitable transformations, we remove the derivatives of the function and obtain some derivative-free family of Halley and Householder type iterative methods. We also give several examples to illustrate the efficiency of these methods. Comparison with other similar method is also given. These new methods can be considered as alternative to the developed derivative-free methods. This technique can be used to suggest a wide class of new iterative methods for solving nonlinear equations.

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1. Introduction

Finding the approximate solution of the nonlinear equation $f(x)=0$ is one of the basic problems and frequently occurs in scientific work of various fields. Due to the higher order of the equation and the involvement of the transcendental functions, analytical methods for obtaining the exact root cannot be employed and therefore, it is only possible to obtain approximate solutions by relying on numerical methods based on iteration procedure [1-19]. If we come across a problem that the function $f(x)$ is not known explicitly or the derivatives of the fuction are difficult to compute, then a method that uses only computed values of the function is more appropriate.

Bisection method, Secent method and Regula-falsi method are the classical numerical methods for solving nonlinear equation without using any derivative of the function [19]. These are the basic methods but have slow convergence toward the solution having some additional drawbacks.

Newton's method, which is straightforward and converges quadratically [19] is probably the best known and most extensively used algorithm which includes the derivative of the function. However, Steffensen's method [4, 19]

$$x_{n+1}=x_n-\frac{\left[f(x_n)\right]^2}{f(x_n+f(x_n))-f(x_n)}, \qquad n=0,1,2,\cdots, \tag{1}$$

is variation on Newton's method which do not employ the derivative of the function. In this method the derivative is approximated by the forward difference scheme. Steffensen's method has same order of convergence as Newton's method.

Based on the approximation of the first derivative, in this paper we construct some derivative-free iterative methods for solving nonlinear equations. We also modify the well known Halley and Householder methods as derivative-free iterative methods with the cubic order of convergence for solving nonlinear equations.

2. Construction of iterative methods

In this section, we derive some new iterative methods for solving nonlinear equations without using the derivative of the function. We use approximation of first derivative of the function to obtain such type of iterative methods.

Let us approximate the first derivative of the function $f(x)$ by

$$f'(x)\approx\frac{f(x+bf(x))-f(x)}{bf(x)}, \text{ where } b\in R \text{ and } b\neq 0. \tag{2}$$

For every value of x during the iteration process. We use this approximation in some efficient methods which involve the first derivative of the function and obtain some alternative derivative-free methods.

Now we use the relation (2) in well known Newton method, we obtain the following derivative-free iterative method for solving nonlinear equation as:

Algorithm 2.1. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1}=x_n-\frac{b\left[f(x_n)\right]^2}{f\left(x_n+bf(x_n)\right)-f(x_n)}, \quad n=0,1,2,\cdots.$$

For $b=1$, well known Steffensen's method [5, 19] can be produced from Algorithm 2.1.

Noor [10] suggested the iterative method for solving nonlinear equation which involves the first derivative of the function as:

Algorithm 2.2. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - \mu f(x_n)}, \quad n = 0, 1, 2, \cdots.$$

Now we use (2) in Algorithm 2.2 and obtain a derivative-free method as:

Algorithm 2.3. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{b[f(x_n)]^2}{f(x_n + b\,f(x_n)) - f(x_n) - \mu b[f(x_n)]^2},$$

$$n = 0, 1, 2, \cdots.$$

If $b = 1$, and $\mu = 0$, then the Algorithm 2.2 reduces to the well known Steffensen's method [3, 19].

Noor and Shah [11] used variational iteration technique and suggested a third-order iterative method for solving nonlinear equation is given below which involves the derivative of the function.

Algorithm 2.4. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n) - \mu f(x_n)}, \; n = 0, 1, 2, \cdots,$$

Now we modify by applying (2) and obtain third-order derivative-free method as:

Algorithm 2.5. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{b[f(x_n)]^2}{f(x_n + bf(x_n)) - f(x_n)},$$

$$x_{n+1} = x_n - \frac{bf(x_n)f(y_n)}{f(x_n + b\,f(x_n)) - f(x_n) - \mu b[f(x_n)]^2},$$
$$n = 0, 1, 2, \cdots,$$

which is a third-order derivative-free method for solving nonlinear equations.

Now we want to modify some well known third-order iterative methods which involve second derivative of the function during iteration process at each step. We consider Halley method [1, 19] and Householder method [1, 8, 19].

Algorithm 2.6. [1, 19]. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}, \quad n = 0, 1, 2, \cdots,$$

and Householder method [1, 8, 19]

Algorithm 2.7. [1, 8, 19]. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x_n)}{2[f'(x_n)]^3}, \; n = 0, 1, 2, \cdots,$$

which involve the second derivative also. For these methods we approximate the first and second derivative of the function $f(x)$ at the current iteration x_n by

$$f'(x_n) \approx \frac{f(x_n + bf(x_n)) - f(x_n - bf(x_n))}{2b\,f(x_n)}, \tag{3}$$

$$f''(x_n) \approx \frac{f(x_n + bf(x_n)) - 2f(x_n) + f(x_n - bf(x_n))}{b^2[f(x_n)]^2}, \tag{4}$$

where $b \in R$ and $b \neq 0$.

to obtain the derivative free methods as:

Algorithm 2.8. For a given x_0, find the approximate solution x_{n+1} by the iterative schemes

$$x_{n+1} = x_n - \frac{2b[f(x_n)]^2[f(x_n + bf(x_n)) - f(x_n - bf(x_n))]}{[f(x_n + bf(x_n)) - f(x_n - bf(x_n))]^2 - 2f(x_n)f(x_n + bf(x_n)) + 4[f(x_n)]^2 - 2f(x_n)f(x_n - bf(x_n))},$$

Algorithm 2.9. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme:

$$x_{n+1} = x_n - \frac{2b[f(x_n)]^2}{[f(x_n + bf(x_n)) - f(x_n - bf(x_n))]} - \frac{4b[f(x_n)]^3[f(x_n + bf(x_n)) - 2f(x_n) + f(x_n - bf(x_n))]}{[f(x_n + bf(x_n)) - f(x_n - bf(x_n))]^3}$$

Remark 2.1. Algorithm 2.1, Algorithm 2.3 and Algorithm 2.5 modified above have at least second and third-order convergence respectively for all values of μ. If we take $\mu = 0, \; \mu = \frac{1}{2}, \; \mu = 1, \cdots,$ in above derived methods, we can obtain various classes of iterative methods for solving nonlinear equations.

Remark 2.2. It is important to state that never opt such a value of μ which makes the denominator zero. It is also essential that sign of μ should be selected so as to keep the denominator largest in magnitude in above derived Algorithms.

3. Convergence analysis

In this section, we consider the convergence criteria of the iterative method Algorithm 2.9 developed in section 2. All other methods can be studied for the convergence on the same patron.

Theorem 3.1. Assume that the function $f: D \subset R \to R$ for an open interval D has a simple root $p \in D.$ Let $f(x)$ be smooth sufficiently in some neighborhood of the root and then the Algorithm 2.9 has third order convergence.

Proof. Let p be a simple root of $f(x)$. Since f is sufficiently differential, then expanding $f(x_n),\ f\left(x_n - bf(x_n)\right)$ and $f\left(x_n + bf(x_n)\right)$ in Taylor's series about $p,$ we get

$$f(x_n) = c_1\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)\right], \tag{5}$$

$$f\left(x_n - bf(x_n)\right) = c_1[(1-bc_1)e_n + (c_2 + 3bc_1c_2 + c_1^2c_2b^2)e_n^2 + (4bc_1c_3 + c_3 + 3c_1^2c_3b^2 + c_1^3c_3b^3 + 2c_2^2c_1b + 2c_2^2c_1^2b^2)e_n^3 + O(e_n^4)]. \tag{6}$$

and

$$f\left(x_n + bf(x_n)\right) = c_1[(1+bc_1)e_n + (c_2 - 3bc_1c_2 + c_1^2c_2b^2)e_n^2 + (-4bc_1c_3 + c_3 + 3c_1^2c_3b^2 - c_1^3c_3b^3 - 2c_2^2c_1b + 2c_2^2c_1^2b^2)e_n^3 + O(e_n^4)]. \tag{7}$$

Where

$$c_k = \frac{1}{k!}\frac{f^{(k)}(p)}{f'(p)}, \quad k = 2,3,\cdots, \quad c_1 = f'(p), \qquad \text{and}$$

$$e_n = x_n - p.$$

Now using (6) and (7), we obtain

$$f(x_n + bf(x_n)) - f(x_n - bf(x_n)) = 2bc_1^2e_n + 6bc_1c_2e_n^2 + (8bc_1^2c_3 + 2b^3c_1^4c_3 + 4bc_1^2c_2^2)e_n^3 + O(e_n^4). \tag{8}$$

Using (5), (6) and (7), we get

$$f(x_n + bf(x_n)) - 2f(x_n) + f(x_n - bf(x_n)) = 2b^2c_1^3c_2e_n^2 + (6b^2c_1^3c_3 + 4b^2c_1^3c_2^2)e_n^3 + O(e_n^4). \tag{9}$$

Now using (5) and (9), we obtain

$$\frac{f(x_n + bf(x_n)) - 2f(x_n) + f(x_n - bf(x_n))}{b^2 f(x_n)} = c_1^2\left[2c_2e_n + (2c_2^2 + 6c_3)e_n^2 + (2c_2^2 + 8c_2c_3 + 12c_4 + 2c_1^2c_4b^2)e_n^3 + O(e_n^4)\right]. \tag{10}$$

Using (5) and (8), we get

$$\frac{2b\left[f(x_n)\right]^2}{\left[f(x_n + bf(x_n)) - f(x_n - bf(x_n))\right]} = e_n - c_2e_n^2 + (2c_2^2 - 2c_3 - c_1^2c_3b^2)e_n^3 + (7c_2c_3 - 3c_4 + b^2c_1c_2c_3 - 4b^2c_1^2c_4 - 4c_2^3)e_n^4 + O(e_n^5). \tag{11}$$

Using (5), (8) and (9), we get

$$\frac{4b\left[f(x_n)\right]^3\left[f(x_n + bf(x_n)) - 2f(x_n) + f(x_n - bf(x_n))\right]}{\left[f(x_n + bf(x_n)) - f(x_n - bf(x_n))\right]^3} = \tag{12}$$

$$c_2e_n^2 + (3c_3 - 4c_2^2)e_n^3 + (19c_2c_3 + 6c_4 - 3b^2c_1^2c_2c_3 - 4b^2c_1^2c_4 + 13c_2^3 + b^2c_1^2c_4)e_n^4 + O(e_n^5).$$

Now using (11) and (12), we obtain

$$x_{n+1} = p + (2c_2^2 - c_3 + b^2c_1^2c_3)e_n^3 + (12c_2c_3 - 3c_4 + 2b^2c_1^2c_2c_3 - 9c_2^3 + 3b^2c_1^2c_4)e_n^4 + O(e_n^5). \tag{13}$$

Finally, the error equation is

$$e_{n+1} = (2c_2^2 - c_3 + b^2c_1^2c_3)e_n^3 + (12c_2c_3 - 3c_4 + 2b^2c_1^2c_2c_3 - 9c_2^3 + 3b^2c_1^2c_4)e_n^4 + O(e_n^5). \tag{14}$$

This shows that the Algorithm 2.9 is of third-order. This completes the proof.

4. Numerical results

We now present some examples to illustrate the efficiency of the new developed iterative methods in Tables 1–7. We compare Steffensen's method (SM), with the Algorithm 2.1, Algorithm 2.3, Algorithm 2.5, Algorithm 2.8 and Algorithm 2.9 with different values of μ and b.

We use the following examples for the comparison of the methods.

$$f_1(x) = \sin^2 x - x^2 + 1,$$
$$f_2(x) = x^3 + 4x^2 - 10,$$
$$f_3(x) = x^2 - e^x - 3x + 2,$$
$$f_4(x) = \cos(x) - x,$$

$$f_5(x) = (x-2) - e^{-x},$$
$$f_6(x) = x^3 + 4x^2 + 8x + 8,$$
$$f_7(x) = \sin x - \frac{1}{2}x.$$

Table 1. Examples and comparison of various iterative methods

$f(x)$	x_0	SM $(b=1, \mu=0)$	Algorithm 2.3 $(\mu=1)$ $(b=-1)$	$(\mu=1)$ $(b=1/2)$	$(\mu=1)$ $(b=-1/2)$	$(\mu=1/2)$ $(b=-1)$	$(\mu=1/2)$ $(b=1/2)$	$(\mu=1/2)$ $(b=-1/2)$
f_1	−1	10	4	5	6	5	5	6
f_2	2	18	14	9	6	8	10	5
f_3	−2	38	6	12	13	7	9	10
f_4	1.7	5	5	5	6	5	6	5
f_5	0	7	5	7	6	7	7	7
f_6	−1	9	6	8	9	7	7	7
f_7	−1	7	6	7	6	4	4	6

Table 2. $(\mu=0)$ *Examples and comparison of various iterative methods*

$f(x)$	x_0	SM	Algorithm 2.5 $(b=1)$	$(b=-1)$	$(b=1/2)$	$(b=-1/2)$
f_1	−1	10	4	5	3	4
f_2	2	18	10	6	7	4
f_3	−2	38	21	6	4	5
f_4	1.7	5	3	3	3	3
f_5	0	7	5	3	4	3
f_6	−1	9	6	5	5	3
f_7	−1	7	7	7	5	5

Table 3. $(\mu=1)$ *Examples and comparison of various iterative methods*

$f(x)$	x_0	SM	Algorithm 2.5 $(b=1)$	$(b=-1)$	$(b=1/2)$	$(b=-1/2)$
f_1	−1	10	5	4	5	4
f_2	2	18	10	6	7	7
f_3	−2	38	21	4	4	5
f_4	1.7	5	4	4	3	4
f_5	0	7	5	3	5	3
f_6	−1	9	5	4	4	4
f_7	−1	7	4	4	5	5

Table 4. $(\mu = \frac{1}{2})$ *Examples and comparison of various iterative methods*

$f(x)$	x_0	SM	Algorithm 2.5			
			$(b=1)$	$(b=-1)$	$(b=1/2)$	$(b=-1/2)$
f_1	-1	10	4	4	3	4
f_2	2	18	10	6	7	6
f_3	-2	38	21	5	4	4
f_4	1.7	5	4	4	3	3
f_5	0	7	4	3	4	3
f_6	-1	9	6	5	5	4
f_7	-1	7	4	3	5	4

Table 5. Examples and comparison of various iterative methods

$f(x)$	x_0	SM	Algorithm 2.8			
			$(b=1/2)$	$(b=1/2)$	$(b=1/4)$	$(b=-1/4)$
f_1	-1	10	4	4	3	4
f_2	2	18	6	6	5	4
f_3	-2	38	5	5	4	4
f_4	1.7	5	3	3	3	3
f_5	0	7	4	4	3	3
f_6	-1	9	5	5	4	3
f_7	-1	7	6	6	5	5

Table 6. Examples and comparison of various iterative methods

$f(x)$	x_0	SM	Algorithm 2.9			
			$(b=1/2)$	$(b=1/2)$	$(b=1/4)$	$(b=-1/4)$
f_1	-1	10	5	5	3	4
f_2	2	18	6	6	5	5
f_3	-2	38	5	5	4	4
f_4	1.7	5	3	3	3	3
f_5	0	7	4	4	3	3
f_6	-1	9	4	5	4	4
f_7	-1	7	4	4	4	4

As convergence criteria, we use the following criteria

(i) $|x_n - x_{n-1}| < \varepsilon,$

(ii) $|f(x_n)| < \varepsilon,$

(iii) Maximum number of iterations to be calculated=300, We used $\varepsilon = 10^{-15}$. Displayed in all Tables 1–6 is the number of iterations to approximate the zero (IT) for the methods. The computational results presented in Tables show that for most of the functions we tested, the presented methods are efficient and show better performance as compared with the Steffensen's method. All the results are verified by using Maple software.

5. Conclusion

In this article, we have suggested new derivative-free iterative methods for solving nonlinear equations. These all are derivative-free methods. Well known Halley method and Householder methods are also modified as derivative-free methods having the same order of convergence i.e. cubic. For different values of the parameter μ, we can derive new classes of iterative methods for solving nonlinear equations. This transformation can also be applied for solving systems of nonlinear equations.

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